Chapter - Integral

Key Concepts

Integration is the reverse process of differentiation

e.g. If $\frac{d}{dx}(sinx) = cosx$ then $\int cosxdx = sinx + any\ constant$

(A)Indefinite Integrals

* $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ **n ≠ -1**
* $\int \frac{1}{\sqrt{x}} = 2\sqrt{x} + c$
* $\int a^x dx = \frac{a^x}{\log a} + c$
* $\int \sin x dx = -\cos x + c$
* $\int \operatorname{cosec}^2 x\, dx = -\cot x + c$
* $\int \operatorname{cosec} x . \cot x\, dx = -\operatorname{cosec} x + c$
* $\int \cot x\, dx = \log|\sin x| + c = -\log|\operatorname{cosec} x|$
* $\int \sec x\, dx = \log|\sec x + \tan x| + C = \log\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$
* $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$, if x > a
* $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$, if x > a
* $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, = -\frac{1}{a}\cot^{-1}\frac{x}{a} + C$
* $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2 + a^2}\right| + C$
* $\int 1.dx = x + c$
* $\int \frac{1}{x} dx = \log x + c$
* $\int e^x dx = e^x + c$
* $\int \cos x dx = \sin x + c$
* $\int \sec^2 x\, dx = \tan x + c$
* $\int \sec x . \tan x\, dx = \sec x + c$
* $\int \tan x\, dx = -\log|\cos x| + c = \log|\sec x| + c$
* $\int \operatorname{cosec} x\, dx = \log|\operatorname{cosec} x - \cot x| + C$
* $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$
* $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a} + c = -\cos^{-1}\frac{x}{a} + c$
* $\int \frac{dx}{\sqrt{a^2 + x^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$
* $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right| + C$

* $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$ * $\int \{f_1(x) \pm f_2(x) \pm \ldots\ldots f_n(x)\}dx$ * $\int f_1(x)dx \pm \int f_2(x)dx \pm \ldots\ldots \pm \int f_n(x)dx$	$\int \lambda f(x)\,dx = \lambda \int f(x)\,dx + C$

(B) General Properties of Definite Integrals.

* $\int_a^b f(x)\,dx$ = F(b) – F(a), where F(x) = $\int f(x)\,dx$ * $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ * $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ * $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is an even function of } x. \\ 0 & \text{if } f(x) \text{ is an odd function of } x \end{cases}$	* $\int_a^b f(x)\,dx = \int_a^b f(t)\,dx$ * $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ * $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ * $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x). \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$

(C) Integration by parts

* $\int u.v\,dx = u.\int v.dx - \int\left[\int v.dx\right]\frac{du}{dx}.dx$ (Here u is considered as first function and *v* is considered as second function)

Note:

(1) We can use the order ***ILATE*** for sequencing the first function and second function, where
I = Inverse Trigonometric functions
L = Logarithmic functions
A = Algebraic functions
T = Trigonometric functions
E= Exponential functions

(2) If the Integrant contains only one function, we take that function as the first function and 1 as the second function.

(D) Integral as a limit of a sum

$\int_a^b f(x)dx = \lim_{h\to 0} h\,\{ f(a) + f(a+h) + \cdots + f(a+\overline{n-1})h \}$ where $nh=b-a$

$$\int_a^b f(x)dx = (b-a) \lim_{n\to\infty} \frac{1}{n} \{ f(a) + f(a+h) + \cdots + f(a+\overline{n-1})h \}$$

(E)Special types of Integration

(1) Evaluation of Integrals of the form $\int \frac{1}{ax^2+bx+c}dx$ or $\int \frac{1}{\sqrt{ax^2+bx+c}}dx$

Express $ax^2 + bx + c$ as a sum or difference of squares of two $x^2 \pm a^2$

(2)Evaluation of Integrals of the form $\int \frac{px+q}{ax^2+bx+c}dx$ or $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Express $px + q = A\frac{d}{dx}(ax^2 + bx + c) + B$ which reduces to any one of standard form

(3)Evaluation of Integrals of the form

$$\int \frac{1}{asin^2x + bcos^2x}, \int \frac{1}{a + bcos^2x}, \int \frac{1}{a + bsin^2x}, \int \frac{1}{asin^2x + bcos^2x + c}, \int \frac{1}{(asinx + bsinx)^2}$$

To evaluate these types of integrals we have to do the following:

(i) Divide both numerator and denominator by cos^2x

(ii) Put $tanx = t$ and simplify which reduces the integral of the form $\int \frac{1}{at^2+bt+c}dx$

(4)Evaluation of Integrals of the form

$\int \frac{1}{asinx+bcosx}dx$, $\int \frac{1}{a+bcosx}dx$, $\int \frac{1}{a+bsinx}dx$, $\int \frac{1}{asinx+bcosx+c}dx$

To evaluate these types of integrals we have to do the following:

(i) Put $sinx = \frac{2tan\frac{x}{2}}{1+tan^2\frac{x}{2}}$, $cosx = \frac{1-tan^2\frac{x}{2}}{1+tan^2\frac{x}{2}}$

(ii) Replace $1 + tan^2\frac{x}{2} = sec^2\frac{x}{2}$

(iii) Put $tanx = t$ and simplify it.

(5)Evaluation of Integrals of the form

$$\int e^x\{f(x) + f'(x)\}dx = e^x f(x) + c$$

(6) Evaluation of Integrals of the form

S. No.	Form of Integrals	Substitution
1	$\int \sqrt{ax^2+bx+c}\ dx$	Convert to the form $\sqrt{x^2-a^2}$ or $\sqrt{x^2+a^2}$ or $\sqrt{a^2-x^2}$
2	$\int (px+q)\sqrt{ax^2+bx+c}$	Convert $(px+q)$= A $\frac{d(ax^2+bx+c}{dx}$ + B

(7) To evaluate the integrals of the form of rational function:

S. No.	Form of rational function	Form of partial fraction
1	$\frac{px+q}{(x-a)(x-b)}$	$\frac{A}{x-a}+\frac{B}{x-b}$
2	$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$
3	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
4	$\frac{px^2+qx+r}{(x-a)^2(x-c)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-c}$
5	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{(x^2+bx+c)}$

Note; (1) If degree of Numerator polynomial ≥ degree of Denominator polynomial then first we divide Numerator by Denominator and express it in the form of Quotient + Proper fraction

e.g. $\frac{x^3}{x^2-5x+6}=(x+5)+\frac{19x-30}{x^2-5x+6}=(x+5)+\frac{19x-30}{(x-2)(x-3)}$ and simplify it.

(ii) **Thumb Rule :** For $\frac{px+q}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b}$

First we put x = a to find numerator of (x – a) and hide the (x-a) put x = b to find numerator of (x – b) $i.e\quad A=\frac{pa+q}{a-b}$, $B=\frac{pb+q}{b-a}$

(A(x-b) + B(x- a) = px+q$i.e\ A=\frac{pa+q}{a-b}$, $B=\frac{pb+q}{b-a}$ This is known as Thumb Rule

Important Board Questions

(A) Indefinite Integrals

1. Evaluate $\int (5x+3)/\sqrt{(x^2+4x+10)}\ dx$

Solution. . 5x + 3 = A d/dx(x^2 +4x +10) + B

A(2x + 4) + B

= A = 5/2 , B = -7

I = 5/2 ∫ 2x + 4/ √(x^2 +4x +10) dx – 7∫ dx/ √(x2 +4x +10)

Getting I = ∫5/2 √(x^2 +4x +10) - 7 ∫ dx/ √$(x+2)^2$ +$(\sqrt{6})^2$

Getting I = ∫5/2 √(x2 +4x +10) - 7 log|x + 2 + √(x2 +4x +10|

2. Evaluate the following: $\int \frac{x^4}{(x-1)(x^2+1)} dx$.

Solution:-

$$\int \frac{x^4}{(x-1)(x^2+1)} dx$$

$$\frac{x^4}{(x-1)(x^2+1)} = (x+1) + \frac{1}{2(x-1)} - \frac{2x}{4(x^2+1)} - \frac{1}{2(x^2+1)}$$

$$\therefore \int \frac{x^4}{(x-1)(x^2+1)} dx = \int (x+1)dx + \int \frac{1}{2(x-1)} dx - \int \frac{2x}{4(x^2+1)} dx - \int \frac{1}{2(x^2+1)} dx$$

$$= \frac{x^2}{2} + x + \frac{1}{2}\log(x-1) - \frac{1}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1} x + C$$

3. Evaluate $\int \frac{(x-4)\ ex}{(x-2)^3} dx.$

Solution:-
I = ∫ (x – 2 - 2) $e^x/(x-2)^3$ dx = ∫e^x [$1/(x-2)^2 - 2/(x-2)^3$] dx

Taking f(x) = $1/(x-2)^2$ & f'(x) = – $2/(x-2)^3$

using formula ∫e^x (f(x) + f'(x)) dx = e^x f(x) + C => I = $e^x / (x-2)^2$+ C

4. ***Evaluate*** : $\int \left[\log(\log x) + \frac{1}{(\log x)^2}\right] dx$

Sol : let $\log x = t$, $x = e^t$, $dx = e^t dt$

$\int \left[\log t + \frac{1}{t^2}\right] e^t\ dt$

$\int e^t\left[\left(\log t+\frac{1}{t}\right)-\left(\frac{1}{t}-\frac{1}{t^2}\right)\right]dt$

$e^{t\, logt}-\frac{1}{t}.e^t+c.$

$x\ log(log\ x)-\frac{x}{\log x}+c$

5. Evaluate $\int e^x\frac{(x^2+1)}{(x+1)^2}dx$

$I=\int\left(\frac{x^2-1+1+1}{(x+1)^2}\right)e^x dx$

$=\int\left(\frac{x^2-1}{(x+1)^2}+\frac{2}{(x+1)^2}\right)e^x dx$

$=\int\left(\frac{x-1}{x+1}+\frac{2}{(x+1)^2}\right)e^x dx$

$take\ f(x)=\frac{x-1}{x+1}$

$Then\ f'(x)=\frac{2}{(x+1)^2}$

Now it is of the form $\int e^x(f(x)+f'(x))dx$

Hence I= $e^x f(x)+c=e^x\left(\frac{x-1}{x+1}\right)+c$

(A) Definite Integrals

(1) Evaluate $\int_0^{\frac{\pi}{2}}\log(sinx).dx$

$Let\ I=\int_0^{\frac{\pi}{2}}\log(sinx).dx$

$$I = \int_0^{\frac{\pi}{2}} \log \sin\left(\frac{\pi}{2} - x\right) dx = \int_0^{\frac{\pi}{2}} \log \cos x\, dx$$

$$2I = \int_0^{\frac{\pi}{2}} \log \cos x\, dx + \int_0^{\frac{\pi}{2}} \log \cos x\, dx$$

$$= \int_0^{\frac{\pi}{2}} \log \sin 2x\, dx - \int_0^{\frac{\pi}{2}} \log 2\, dx$$

$$= \frac{1}{2}\int_0^{\pi} \log \sin t\, dt - \frac{\pi}{2}\log 2$$

$$= \frac{1}{2} \times 2\int_0^{\frac{\pi}{2}} \log \sin x\, dx - \frac{\pi}{2}\log 2$$

$$\Rightarrow I = -\frac{\pi}{2}\log 2$$

2. Evaluate $\int_0^{\pi/4}$ □log(1+tanx) dx.

Let I $=\int_0^{\pi/4} \log(1+tanx)\, dx$

$= \int_0^{\pi/4} \log\left\{1 + \tan\left(\frac{\pi}{4} - x\right)\right\} dx$

I $=\int_0^{\pi/4}$ (B) $\log\left\{1 + \frac{1-tanx}{1+tanx}\right\} dx$

$=\int_0^{\pi/4} \log\left(\frac{2}{1+tanx}\right) dx$

$=\int_0^{\pi/4} \log 2dx - \int_0^{\pi/4} \log(1+tanx)\, dx$

I =log2$\int_0^{\pi/4} 1dx$- I

2I = log 2 $\left(\frac{\pi}{4} - 0\right)$

I = $\frac{\pi}{8}$log 2

3.Evaluate $\int_1^4 [|x-1| + |x-2| + |x-3|]dx$

Solution:-

$$f(x) = \left[|x-1| + |x-2| + |x-3|\right] = \begin{cases} 4-x, & if\ 1 < x < 2 \\ x, & if\ 2 < x < 3 \\ 3x-6, & if\ 3 < x < 4 \end{cases}$$

$$We\ know\ that,\ \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx + \int_d^b f(x)dx,\ \text{where}\ a < c < d < b$$

$$\int_1^4 \left[|x-1| + |x-2| + |x-3|\right]dx = \int_1^2 \left[|x-1| + |x-2| + |x-3|\right]dx + \int_2^3 \left[|x-1| + |x-2| + |x-3|\right]dx + \int_3^4 \left[|x-1| + |x-2| + |x-3|\right]dx$$

$$= \int_1^2 (4-x)dx + \int_2^3 xdx + \int_3^4 (3x-6)dx$$

$$= \frac{19}{2}$$

4. Evaluate $\int_0^{\frac{\pi}{2}} \frac{x + sinx}{1 + cosx}$ dx

Solution:-

$$\int_0^{\frac{\pi}{2}} \frac{x + \sin x}{1 + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{x + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} dx \quad ---------(1)$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} x\sec^2\frac{x}{2}dx + \int_0^{\frac{\pi}{2}} \tan\frac{x}{2}dx \quad ---------(1)$$

By solving further we get given =

$$\frac{\pi}{2}\tan\frac{\pi}{4} = \frac{\pi}{2} \quad ----------------(2)$$

5. Evaluate $\int_0^{\pi} \frac{\text{x tan x}}{\text{Sec x.Cosec x}}$ dx

. Let I = $\int_0^{\pi} \frac{\text{x tan x}}{\text{Sec x.Cosec x}}$ dx = $\int_0^{\pi} x.\sin^2 x\ dx$ => I = $\int_0^{\pi} (\pi - x).\sin^2 x\ dx$

=> 2I = $\int_0^{\pi} x.\sin^2 x\ dx$ Solving to get I = $\pi^2/4$

6.

$$\int_0^{\frac{\pi}{2}} (2\log\sin x - \log\sin 2x)dx$$

$\int$

Solution:

$$I=\int_0^{\frac{\pi}{2}}\log\left(\frac{\sin^2 x}{2\sin x\cos x}\right)dx$$

$$I=\int_0^{\frac{\pi}{2}}\log\left(\frac{\tan x}{2}\right)dx$$

$$I=\int_0^{\frac{\pi}{2}}\log\left(\frac{\tan\left(\frac{\pi}{2}-x\right)}{2}\right)dx = I=\int_0^{\frac{\pi}{2}}\log\left(\frac{\cot x}{2}\right)dx$$

$$2I=\int_0^{\frac{\pi}{2}}\log\left(\frac{\tan x}{2}\right)\left(\frac{\cot x}{2}\right)dx=\int_0^{\frac{\pi}{2}}\log\left(\frac{1}{4}\right)dx$$

$$I=\frac{\pi}{4}\log\left(\frac{\pi}{4}\right)$$

HOTS

2.

1. Evaluate $\int_{-2}^{2}\frac{x^2}{1+5^x}dx$

Solution: let I $=\int_{-2}^{2}\frac{x^2}{1+5^x}dx$ ... (i)

$\int_{-2}^{2}\frac{(2+(-2)-x)^2}{1+5^{(2+(-2)-x)}}dx$Using $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)dx, & \text{if } f(x) \text{ is an even function of } x. \end{cases}$

I $=\int_{-2}^{2}\frac{x^2 5^x}{1+5^x}dx$ ... (ii)

Adding equation (I and (ii) , we get

2I $=\int_{-2}^{2}x^2dx=\frac{16}{3}$

I $=\frac{8}{3}$

2. Evaluate $\int_0^1 \cot^{-1}(1-x+x^2)\,dx$

Let $I=\int_0^1\cot^{-1}(1-x+x^2)\,dx=\int_0^1\tan^{-1}\frac{1}{1-x+x^2}$ dx

$\int_0^1 \tan^{-1}\frac{x+(1-x)}{1-x(1-x)}\,dx = \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}(1-x)dx = 2\int_0^1 \tan^{-1}x dx$

$=2\{[x\tan^{-1}x]_0^1 - \int_0^1 \frac{x}{1+x^2}dx\ \}= 2\pi/4\ - (\log|1+x^2|)_0^1 = \frac{\pi}{2} - \log 2.$

3. Evaluate $\int \frac{x^2}{(xsinx+cosx)^2}$ **dx** **Ans.** $\frac{sinx-xcosx}{xsinx+cosx}$ **+ C**

4. Evaluate $\int \frac{1}{\sqrt{sin^3\ x\ sin(x+\alpha)}}$ **dx** **Ans. -2cosec**$\alpha\sqrt{(cos\alpha + cotx\ sin\alpha)}$ **+ C**

5. Evaluate $\int \frac{sinx-x\ cosx}{x(x+sinx)}$ **dx** **Ans** $\log|x| - log\ |x+sinx| + c$

6. Evaluate $\int_0^{\frac{3}{2}}|x\ cos\pi x|\ dx$ **Ans**$\frac{5}{2\pi} - \frac{1}{\pi^2}$

7. Prove that :$\int_{-a}^{a}\sqrt{\frac{a-x}{a+x}}\ dx = a\pi$

8. Evaluate $\int \sqrt{tanx}\ dx$

Ans$\frac{1}{\sqrt{2}}\tan^{-1}(\frac{tanx-1}{\sqrt{2tanx}}) + \frac{1}{2\sqrt{2}}log\left|\frac{tanx-\sqrt{2}\ tanx+1}{tanx+\sqrt{2}\ tanx+1}\right|$

9. Evaluate $\int_{-1}^{2}|x^3-x|dx$ *Ans*: $\frac{11}{4}$

10. Evaluate $\int e^{2x}\ sinx\ dx$, *Ans*: $\frac{e^{2x}}{5}(2sinx - cosx) + c$