MATRICES AND DETERMINANTS

KEY CONCEPTS

MATRIX :

A matrix is a rectangular array of m × n numbers arranged in m rows and n columns.

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots \cdots \cdots \cdots & a_{1n} \\ a_{21} & a_{22} \cdots \cdots \cdots \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & a_{m2} \cdots \cdots \cdots & a_{mn} \end{bmatrix}_{m \times n}$$ OR $A = [a_{ij}]_{m \times n}$, where **i** = 1, 2,…., m ; **j** = 1, 2,….,n.

* **Row Matrix** : A matrix which has one row is called row matrix. $A = [a_{ij}]_{1 \times n}$

* **Column Matrix**: A matrix which has one column is called column matrix. $A = [a_{ij}]_{m \times 1}$.

* **Square Matrix**: A matrix in which number of rows are equal to number of columns, is called a square matrix $A = [a_{ij}]_{m \times m}$

* **Diagonal Matrix** : A square matrix is called a Diagonal Matrix if all the elements, except the diagonal elements are zero.

* **Scalar Matrix**: A square matrix is called scalar matrix it all the elements, except diagonal elements are zero and diagonal elements are equal.

$A = [a_{ij}]_{n \times n}$, where $a_{ij} = 0$,**i**≠ j.

$a_{ij} = \alpha$,**i** = j.

* **Identity or Unit Matrix**: A square matrix in which all the non diagonal elements are zero and diagonal elements are unity is called identity or unit matrix.

* **Null Matrix**: A matrix in which all element are zero.

* **Equal Matrices**: Two matrices are said to be equal if they have same order and all their corresponding elements are equal.

* **Transpose of matrix**: If A is the given matrix, then the matrix obtained by interchanging the rows and columns is called the transpose of a matrix.

* **Properties of Transpose**:

If A & B are matrices such that their sum & product are defined, then

(i). $(A^T)^T = A$

(ii). $(A + B)^T = A^T + B^T$

(iii). $(KA^T) = KA^T$ where K is a scalar.

(iv). $(AB)^T = B^T A^T$

(v). $(ABC)^T = C^T B^T A^T$.

* **Symmetric Matrix**: A square matrix is said to be symmetric if

$A = A^T$ i.e. If $A = [a_{ij}]_{m \times m}$, then $a_{ij} = a_{ji}$ for all i, j. Also elements of the symmetric matrix are symmetric about the main diagonal.

For example $A = \begin{bmatrix} \sqrt{3} & 2 & 3 \\ 2 & -1.5 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ is a symmetric matrix as $A' = A$

* **Skew symmetric Matrix** : A square matrix is said to be skew symmetric if $A^T = -A$.

If $A = [a_{ij}]_{m \times m}$, then $a_{ij} = -a_{ji}$ for all i, j.

For example, the matrix $B = \begin{bmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{bmatrix}$ is a skew symmetric matrix as $B' = -B$

Multiplication of a matrix by a scalar:

IfA = [aij] m × n is a matrix and k is a scalar, then kA is another matrix which is obtained by multiplying each element of A by the scalar k.

e.g. $A = \begin{pmatrix} 1 & 3 & 5 \\ 5 & 9 & 5 \\ 8 & 2 & 4 \end{pmatrix}$ then $2A = \begin{pmatrix} 2 & 6 & 10 \\ 10 & 18 & 10 \\ 16 & 4 & 8 \end{pmatrix}$

If $A = [a_{ij}]_{m \times m}$, then $a_{ij} = -a_{ji}$ for all i, j.

***Product of matrices:** If A & B are two matrices, then product AB is defined, if no. of columns of A = no. of rows of B. Let $A = [a_{ij}]_{m \times n}$, and $B = [b_{jk}]_{n \times p}$ then AB is the matrix C of the order mxp. To get $(i.k)^{th}$ element c_{ik} of the matrix C we take i th row of A and k th column of B, multiply them elementwise and take the sum of all these products

Example : $A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$; $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$

$AB = \begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix}$

$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

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NOTE :-

(i) Product of matrices is not commutative. i.e. $AB \neq BA$.

(ii) Product of matrices is associative. i.e $A(BC) = (AB)C$

(iii) Product of matrices distributes over addition.

$A(B + C) = AB + AC$

Determinants :

To every square matrix we can assign a number called its determinant

If $A = [a_{11}]$, det. $A = |A| = a_{11}$.

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $|A| = a_{11}a_{22} - a_{21}a_{12}$.

*Properties :

(i) The value of a determinant remains unchanged if its rows and columns are interchanged.

(ii) if any two rows/ cols. are interchanged , then sign of the determinant changes.

(iii) If two rows/ columns of a determinant are identical, value of the determinant is zero.

(iv) If all the elements of a row/ column of a determinant are multiplied by a constant k, then its value gets multiplied by k. i.e. $|kA| = k|A|$

(v) If elements of any one column (or row) are expressed as sum of two elements each, then determinant can be written as sum of two determinants.

(vi) If A & B are square matrices of same order, then $|AB| = |A|\,|B|$

Minors and Cofactors:

Definition : Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its ith row and jth column in which element a_{ij} lies. Minor of an element aij is denoted by M_{ij}.

Definition : Cofactor of an element a_{ij} , denoted by A_{ij} is defined by $A_{ij} = (-1)^{i+j}M_{ij}$, where M_{ij} is minor of a_{ij}.

***Singular matrix:** A square matrix 'A' of order 'n' is said to be singular, if $|A| = 0$.

* **Non -Singular matrix**: A square matrix 'A' of order 'n' is said to be non-singular, if $|A| \neq 0$.

***Adjoint of matrix** :

If $A = [a_{ij}]$ be a n-square matrix then transpose of cofactors of element of matrix A, is called the adjoint of A.

$$\text{Adj. } A = [A_{ij}]^T.$$

NOTE :- A(Adj.A) = (Adj. A)A = | A| I.

Invertible Matrix :- A Matrix A is said to be invertible if there exists another matrix B such that AB = BA = I

***Inverse of a matrix** : Inverse of a square matrix A exists, if A is non-singular.

$$A^{-1} = \frac{1}{|A|}\text{Adj.}A$$

NOTE : $|adjA| = |A|^{n-1}$, where 'n' is the order of the matrix A.

***System of Linear Equations**:

$a_1x + b_1y + c_1z = d_1.$

$a_2x + b_2y + c_2z = d_2.$

$a_3x + b_3y + c_3z = d_3.$

In Matrix form the above equations can be written as :-

$$\begin{bmatrix} a_1 & b_2 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

i.e. AX = B

$\Rightarrow X = A^{-1}B$; $\{|A| \neq 0\}$.

***Criteria of Consistency.**

(i) If $|A| \neq 0$, then the system of equations is said to be consistent & has a unique solution .

(ii) If $|A| = 0$ and (adjA)B = 0, then the system of equations is consistent and has infinitely many solutions.

(iii) If $|A| = 0$ and (adj. A)B$\neq$ 0, then the system of equations is inconsistent and has no solution

Important Board Questions

1. Construct a 2x2 matrix $A = [a_{ij}]$ whose elements are given by $a_{ij} = \frac{i}{j}$

Sol. $A = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix}$

2. Evaluate : 3 $\begin{vmatrix} 7 & -2 \\ -10 & 5 \end{vmatrix}$

Sol : 3(35 – 20) = 3 x 15 = 45

3. If A is a Square matrix of order 3 and $|A| = 6$ then $|2A| =$ ____________

Ans. $|A| = 6$ and O(A) = 3

$$\therefore |2A| = 2^3|A| = 8 \times 6 = 48$$

4. If A is a Sq matrix of order 3 and $|A| = 3$ then $|A(adj A)| =$ ____________.

Ans: O(A)=3 and $|A| = 3$; $\therefore |A\ (adjA)| = |A|^n = 3^3 = 27.$

QUESTIONS (1 MARKS)

1Find x if $\begin{bmatrix} 5 & 3x \\ 2y & z \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 12 & 6 \end{bmatrix}$

2. For what value of a, $\begin{bmatrix} 2a & -1 \\ -8 & 3 \end{bmatrix}$ is a singular matrix?
3. What is the value of $|3I_3|$?
4. For what value of K, the matrix $\begin{bmatrix} 2-k & 3 \\ -5 & 1 \end{bmatrix}$ is not invertible?

5. If A is a matrix of order 2x3 and B is a matrix of order 3x5 what is the order of matrix $(AB)^T$

6. Find x if $\begin{vmatrix} 3 & 4 \\ -5 & 2 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ -5 & 3 \end{vmatrix}$

7. If A is a sq matrix of order 3 such that $|adj\,A| = 64$. Find $|A^T|$.

8. If A is a square matrix satisfying $A^2 = I$, then what is the A^{-1} ?

9. Evaluate : $3\begin{vmatrix} 7 & -2 \\ -10 & 5 \end{vmatrix}$

10. Find the value of x : $\begin{vmatrix} x & 4 \\ 2 & 2x \end{vmatrix} = 0$.

11. Evaluate $\begin{vmatrix} 2cosx & -2sinx \\ sinx & cosx \end{vmatrix}$

12. Find x, $\begin{vmatrix} x & x \\ 1 & x \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$.

SOLUTIONS

1. As $\begin{bmatrix} 5 & 3x \\ 2y & z \end{bmatrix} = \begin{bmatrix} 5 & 12 \\ 6 & 4 \end{bmatrix}$

$$\therefore 3x = 12 \Rightarrow x = 4 \; and \; 2y = 6 \Longrightarrow y = 3 \; and \; z = 4$$

2. Given matrix is singular $\therefore \begin{vmatrix} 2a & -1 \\ -8 & 3 \end{vmatrix} = 0$

$$\therefore 6a - 8 = 0 \Longrightarrow a = \frac{4}{3}$$

3. $|3I_3| = 3^3 = 27$

4. The given matrix is not invertible if

$$\begin{vmatrix} 2-k & 3 \\ -5 & 1 \end{vmatrix} \neq 0 \Rightarrow -k+17 \neq 0 \Rightarrow k \neq 17$$

5. AB is a matrix of order 2x5 ⇒ $(AB)^T$ is a matrix of order 5x2.

6. As $\begin{vmatrix} 3 & 4 \\ -5 & 2 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ -5 & 3 \end{vmatrix}, \Rightarrow 26 = 6x + 20 \Rightarrow x = 1$

7. As $|adjA| = |A|^{n-1}, \therefore |A|^{3-1} = 64 \Rightarrow |A| = 8$

As $|A| = |A^T| \Rightarrow |A^T| = 8$

8. As $A^2 = I \Rightarrow A^2A^{-1} = IA^{-1} \Rightarrow A(AA^{-1}) = A^{-1}$

$$\Rightarrow AI = A^{-1} \Rightarrow A = A^{-1}$$

9. $3\begin{vmatrix} 7 & -2 \\ -10 & 5 \end{vmatrix} = 3[35-20] = 45$

10. As $\begin{vmatrix} x & 4 \\ 2 & 2x \end{vmatrix} = 0 \Rightarrow 2x^2 - 8 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

11. $A = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix}$

12. $\begin{vmatrix} 2cosx & -2sinx \\ sinx & cosx \end{vmatrix} = 2\cos^2 x + 2\sin^2 x = 2$

13. $\begin{vmatrix} x & x \\ 1 & x \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} \Rightarrow x^2 - x = 2 \Rightarrow x^2 - x - 2 = 0$

$$\Rightarrow (x+1)(x-2) = 0 \Rightarrow x = -1, 2$$

QUESTIONS (4 Marks Questions)

SHOW THAT

1. $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$

2. $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$ $=-(a+b+c)\,[(a-b)^2+(b-c)^2+(c-a)^2]$

3. If x, y, z are different and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix}$ $= 0$, then $1+xyz=0$

4. $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$

5. $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = xyz+xy+yz+zx$

6. Two schools A and B decided to award prizes to their students for three values Honesty (x), punctuality(y) and Obedience(z). School A decided to award a total of rupees 15000 for three values to 4,3 and 2 students respectively, while school B decided to award Rs. 19000 for three values to 5,4 and 3 students respectively. If all the three prizes together amount to Rs. 5000, then

   (i) Represent the above situation by a matrix equation and form linear equation using matrix multiplication.

   (ii) Which value you prefer to be rewarded most and why?

7. Using properties of determinants, prove that

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^2$$

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SOLUTIONS (4 Marks)

1. $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} a(a^2+1) & a^2b & a^2c \\ ab^2 & b(b^2+1) & b^2c \\ c^2a & c^2b & c(c^2+1) \end{vmatrix}$

$= \frac{abc}{abc}\begin{vmatrix} (a^2+1) & a^2 & a^2 \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$

$= \begin{vmatrix} a^2+b^2+c^2+1 & a^2+b^2+c^2+1 & a^2+b^2+c^2+1 \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$

$=(a^2+b^2+c^2+1)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$

$=(a^2+b^2+c^2+1)\begin{vmatrix} 1 & 0 & 0 \\ b^2 & 1 & -1 \\ c^2 & 0 & 1 \end{vmatrix}$

$=(a^2+b^2+c^2+1)$

2. $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & b+c & c+a \end{vmatrix}$

$= 2(a+b+c)\begin{vmatrix} 1 & c+a & a+b \\ 1 & a+b & b+c \\ 1 & b+c & c+a \end{vmatrix}$

$= 2(a+b+c)\begin{vmatrix} 1 & c+a & a+b \\ 0 & b-c & c-a \\ 0 & b-a & c-b \end{vmatrix}$

$= 2(a+b+c)[(b-c)(c-b)-(b-a)(c-a)]$

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$= -(a+b+c)[2a^2+2b^2+2c^2-2ab-2bc-2ca]$
$= -(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$

3. $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix}=0$

$$or \begin{vmatrix} x-y & x^2-y^2 & x^3-y^3 \\ y-z & y^2-z^2 & y^3-z^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$$

$$or (x-y)(y-z)\begin{vmatrix} 0 & x-z & (x-z)(x+y+z) \\ 1 & y+z & y^2+yz+z^2 \\ z & z^2 & 1+z^3 \end{vmatrix}=0$$

$$or (x-y)(y-z)(x-z)\begin{vmatrix} 0 & 1 & (x+y+z) \\ 1 & y+z & y^2+yz+z^2 \\ z & z^2 & 1+z^3 \end{vmatrix}$$

$or (x-y)(y-z)(x-z)(1+xyz) = 0$ if $1+xyz = 0$

4. $\begin{vmatrix} 1+a^2+b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & 0 & 1+a^2+b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^3 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (1+a^2+b^2)^3$$

5. $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = \begin{vmatrix} 1+x & -x & 0 \\ 1 & y & -y \\ 1 & 0 & z \end{vmatrix}$

= xy + z(xy+x+y) = xy + yz + zx + xyz

6.

school	Honesty	Punctuality	Obedience	Prize(Rs.)
A	4	3	2	15000
B	5	4	3	19000
	1	1	1	5000

(i) Matrix equation is

$$\begin{bmatrix}4 & 3 & 2\\5 & 4 & 3\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}15000\\19000\\5000\end{bmatrix}$$

Linear equations are

4x + 3y + 2z = 15000

5x + 4y + 3z = 19000

X + y + z = 5000

(ii) I would like to reward equally to each value as each value has its own importance in our life.

7.

Apply $C_1 \to C_1 + C_2 + C_3$

then $R_2 \to R_2 - R_1$ *and* $R_3 \to R_3 - R_2$

Six marks questions

Solve using matrix method:

1. x+3y+4z =8, 2x+y+2z =5, 5x+y+z =7

2. 8x+4y+3z =18, 2x+y+z =5, x+2y+z =5

3. $\frac{1}{x}-\frac{1}{y}+\frac{1}{z}=4$, $\frac{2}{x}+\frac{1}{y}+\frac{3}{2}=0$, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$, $x\neq 0$, $y\neq 0$, $z\neq 0$

4. If $A=\begin{bmatrix}1 & -1 & 1\\2 & 1 & -3\\1 & 1 & 1\end{bmatrix}$ find A^{-1} and hence solve.

x+2y+z =4, -x+y+z =0, x-3y+z =2

5. If $A=\begin{bmatrix}-4 & 4 & 4\\-7 & 1 & 3\\5 & -3 & -1\end{bmatrix}$, $B=\begin{bmatrix}1 & -1 & 1\\1 & -2 & -2\\2 & 1 & 3\end{bmatrix}$, find AB and use Hence to solve the following equations:

x-y+z =4, x-2y-2z = 9, 2x+y+3z =1

6. The management committee of a residential colony decided to award some of its members (say x) for honesty, some (say y) for helping others and some other (say z) for supervising the workers to keep the colony neat and clean. The sum of all the awards is 12. Three times the sum of awardees for cooperation and supervision added to two times the number of awardees for honesty is 33. If the sum of awardees for honesty and supervision is twice the number of awardees for helping others, using matrix method, find the number of awardees of each category. Apart from these values, namely, honesty, cooperation and supervision, suggest one more value which the management of the colony must include for awards.

7.Two schools A and B want to award their selected students on the values of sincerity, truthfulness and helpfulness. The school A wants to award Rs. x each, Rs. y each and Rs. z each for the three respective values to 3, 2 and 1 students respectively with a total award money of Rs. 1600. School B wants to spend Rs. 2300 to award its 4, 1 and 3 students on the respective values (by giving the same award money to the three values as before). If the total amount of award for one prize on each value is Rs. 900, using matrices, find the award money for each value. Apart from these three values, suggest one more value which should be considered for award.

8. If $A = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 1 & -3 \\ 0 & 1 & 4 \end{bmatrix}$ find A^{-1} using elementary operation.

SOLUTIONS

1. The given system of equations are

x + 3y + 4z = 8 ,2x + y +2z = 5 and 5x + y + z = 7

This system of equations can be written as

$AX = B$, where $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$

Here $|A| = \begin{vmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{vmatrix} = 11 \neq 0$

Hence A^{-1} exists.

$$adjA = \begin{bmatrix} A_{11} = -1 & A_{12} = 8 & A_{13} = -3 \\ A_{21} = 1 & A_{22} = -19 & A_{23} = 14 \\ A_{31} = 2 & A_{32} = 6 & A_{33} = -5 \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{11}\begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}$$

As $X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}\begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$

$= \frac{1}{11}\begin{bmatrix} 11 \\ 11 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

X = 1, y = 1 , z = 1

2. The given system of equations are

8x + 4y + 3z = 18, 2x + y + z = 5 and x + 2y + z = 5

This system of equations can be written as

AX = B where A = $\begin{bmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$; B = $\begin{bmatrix} 18 \\ 5 \\ 5 \end{bmatrix}$; X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$|A| = \begin{vmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -3 \neq 0$$

$$adjA = \begin{bmatrix} A_{11} = -1 & A_{12} = -1 & A_{13} = 3 \\ A_{21} = 2 & A_{22} = 5 & A_{23} = -12 \\ A_{31} = 1 & A_{32} = -2 & A_{33} = 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 2 & 1 \\ -1 & 5 & -2 \\ 3 & -12 & 0 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{-1}{3}\begin{bmatrix} -1 & 2 & 1 \\ -1 & 5 & -2 \\ 3 & -12 & 0 \end{bmatrix}$$

$AsX = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{3}\begin{bmatrix} -1 & 2 & 1 \\ -1 & 5 & -2 \\ 3 & -12 & 0 \end{bmatrix}\begin{bmatrix} 18 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

x = 1; y = 1 ; z = 2

3. The given system of equations are

$$\frac{1}{x} - \frac{1}{y} + \frac{1}{z} = 4; \frac{2}{x} + \frac{1}{y} + \frac{3}{z} = 0; \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$$

$$Let \frac{1}{x} = u; \ \frac{1}{y} = v; \ \frac{1}{z} = w$$

$$\therefore u - v + w = 4; 2u + v + 3w = 0; u + v + w = 2$$

This system of equations can be written as

AX = B where A=$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -2 \neq 0$$

$$adjA = \begin{bmatrix} A_{11} = -2 & A_{12} = 1 & A_{13} = 1 \\ A_{21} = 2 & A_{22} = 0 & A_{23} = -2 \\ A_{31} = -4 & A_{32} = -1 & A_{33} = 3 \end{bmatrix}^T$$

$$=\begin{bmatrix}-2 & 2 & -4\\ 1 & 0 & -1\\ 1 & -2 & 3\end{bmatrix}$$

$$A^{-1}=\frac{-1}{2}\begin{bmatrix}-2 & 2 & -4\\ 1 & 0 & -1\\ 1 & -2 & 3\end{bmatrix}$$

$$AsX=A^{-1}B \Rightarrow \begin{bmatrix}u\\ v\\ w\end{bmatrix}=\frac{-1}{2}\begin{bmatrix}-2 & 2 & -4\\ 1 & 0 & -1\\ 1 & -2 & 3\end{bmatrix}\begin{bmatrix}4\\ 0\\ 2\end{bmatrix}=\begin{bmatrix}8\\ -1\\ -5\end{bmatrix}$$

u = 8; v = -1; w = -5

$\therefore x=\frac{1}{8}$; y= -1 ; z = $\frac{-1}{5}$

4. Given $A=\begin{bmatrix}1 & -1 & 1\\ 2 & 1 & -3\\ 1 & 1 & 1\end{bmatrix}$

$$\therefore |A|=\begin{vmatrix}1 & -1 & 1\\ 2 & 1 & -3\\ 1 & 1 & 1\end{vmatrix}=10\neq 0$$

$$Hence A^{-1} exists$$

$$adjA=\begin{bmatrix}A_{11}=4 & A_{12}=-5 & A_{13}=1\\ A_{21}=2 & A_{22}=0 & A_{23}=-2\\ A_{31}=2 & A_{32}=5 & A_{33}=3\end{bmatrix}^T$$

$$=\begin{bmatrix}4 & 2 & 2\\ -5 & 0 & 5\\ 1 & -2 & 3\end{bmatrix}$$

$$A^{-1}=\frac{1}{10}\begin{bmatrix}4 & 2 & 2\\ -5 & 0 & 5\\ 1 & -2 & 3\end{bmatrix}$$

Now, the system of equations is

x + 2y + z = 4, -x + y + z = 0; x – 3y + z = 2

The system of given linear can be written as

$$\begin{bmatrix}1 & 2 & 1\\ -1 & 1 & 1\\ 1 & -3 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}4\\ 0\\ 2\end{bmatrix} \quad A^TX=B \text{ where} A=\begin{bmatrix}1 & -1 & 1\\ 2 & 1 & -3\\ 1 & 1 & 1\end{bmatrix}$$

Now, $(A^T)^{-1}(A^TX)=(A^T)^{-1}B \Rightarrow IX=(A^{-1})^TB$

$$\Rightarrow X=\frac{1}{10}\begin{bmatrix}4 & 2 & 2\\ -5 & 0 & 5\\ 1 & -2 & 3\end{bmatrix}^T\begin{bmatrix}4\\ 0\\ 2\end{bmatrix}=\frac{1}{10}\begin{bmatrix}4 & -5 & 1\\ 2 & 0 & -2\\ 2 & 5 & 3\end{bmatrix}\begin{bmatrix}4\\ 0\\ 2\end{bmatrix}=\begin{bmatrix}18/10\\ 4/10\\ 14/10\end{bmatrix}$$

Hence $x=\frac{9}{5}; y=\frac{2}{5}; z=\frac{7}{5}$

5. A= $\begin{bmatrix}-4 & 4 & 4\\-7 & 1 & 3\\5 & -3 & -1\end{bmatrix}$; B= $\begin{bmatrix}1 & -1 & 1\\1 & -2 & -2\\2 & 1 & 3\end{bmatrix}$

AB= $\begin{bmatrix}8 & 0 & 0\\0 & 8 & 0\\0 & 0 & 8\end{bmatrix}$ =8 I_3i.e AB=8 I_3

$A(BB^{-1})=8I_3B^{-1} \Rightarrow A = 8B^{-1}$

$$\therefore B^{-1} = \frac{1}{8}A$$

The system of given equations can be written as

BX = C ,where B = $\begin{bmatrix}1 & -1 & 1\\1 & -2 & -2\\2 & 1 & 3\end{bmatrix}; X = \begin{bmatrix}x\\y\\z\end{bmatrix}; C = \begin{bmatrix}4\\9\\1\end{bmatrix}$

$\Rightarrow B^{-1}(BX) = B^{-1}C$

$\Rightarrow X = B^{-1}C = \left(\frac{1}{8}A\right)C = \frac{1}{8}\begin{bmatrix}-4 & 4 & 4\\-7 & 1 & 3\\5 & -3 & -1\end{bmatrix}\begin{bmatrix}4\\9\\1\end{bmatrix} = \frac{1}{8}\begin{bmatrix}24\\-16\\-8\end{bmatrix} = \begin{bmatrix}3\\-2\\-1\end{bmatrix}$

Hence x = 3, y = -2, z = -1

6. The system of equations is x + y + z = 12

2x + 3y + 3z = 33

x – 2y + z = 0

The system of given equations can be written as

AX = B ,where A = $\begin{bmatrix}1 & 1 & 1\\2 & 3 & 3\\1 & -2 & 1\end{bmatrix}; X = \begin{bmatrix}x\\y\\z\end{bmatrix}; B = \begin{bmatrix}12\\33\\0\end{bmatrix}$

$$|A| = \begin{vmatrix}1 & 1 & 1\\2 & 3 & 3\\1 & -2 & 1\end{vmatrix} = 1(3+6) - 1(2-3) + 1(-4-3) = 3$$

$$A_{11} = 9, A_{12} = 1, A_{13} = -7$$

$$A_{21} = -3, A_{22} = 0, A_{23} = 3$$

$$A_{31} = 0, A_{32} = -1, A_{33} = 1$$

$$adjA = \begin{bmatrix}9 & 1 & -7\\-3 & 0 & 3\\0 & -1 & 1\end{bmatrix}^T = \begin{bmatrix}9 & -3 & 0\\1 & 0 & -1\\-7 & 3 & 1\end{bmatrix}$$

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$$

Now, $X = A^{-1}B = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}\begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 \\ 12 \\ 15 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

For honesty(x)=3 medals

For helping others(y)=4 medals

For supervision(z)=5 medals

Other values are- Truthfulness, sincerity.

7. The system of equations is 3x +2 y + z = 1600
   4x + 1y + 3z = 2300
   x + y + z = 900

The system of given equations can be written as

AX = B

,where A = $\begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; B = \begin{bmatrix} 1600 \\ 2300 \\ 900 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 3(1-3) - 2(4-3) + 1(4-1) = -5$$

$$A_{11} = -2, A_{12} = -1, A_{13} = 3$$

$$A_{21} = -1, A_{22} = 2, A_{23} = -1$$

$$A_{31} = 5, A_{32} = -5, A_{33} = -5$$

$$adjA = \begin{bmatrix} -2 & -1 & 3 \\ -1 & 2 & -1 \\ 5 & -5 & -5 \end{bmatrix}^T = \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{-5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

Now, $X = A^{-1}B = \frac{1}{-5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}\begin{bmatrix} 1600 \\ 2300 \\ 900 \end{bmatrix} = \frac{1}{-5}\begin{bmatrix} -1000 \\ -1500 \\ -2000 \end{bmatrix} = \begin{bmatrix} 200 \\ 300 \\ 400 \end{bmatrix}$

For sincerity (x) =Rs. 200

For truthfulness (y) = Rs. 300

For helpfulness (z) = Rs. 400

Other values are- honesty and helping others.

8. A=IA

$$\begin{bmatrix} 2 & -1 & 1 \\ -2 & 1 & -3 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \rightarrow R_1 + R_3$

$$\begin{bmatrix} 2 & 0 & 5 \\ -2 & 1 & -3 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \rightarrow R_2 + R_1$

$$\begin{bmatrix} 2 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 2 & 0 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} A$$

$R_1 \rightarrow 2R_1 - 5R_3$,$R_2 \rightarrow R_2 - R_3$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 5 & 2 \\ 2 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} A$$

$R_1 \rightarrow \frac{1}{4}R_1, R_3 \rightarrow \frac{1}{2}R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{4} & \frac{5}{4} & \frac{1}{2} \\ 2 & 2 & 1 \\ \frac{-1}{4} & \frac{-1}{4} & 0 \end{bmatrix} A$$

$$A^{-1} = \begin{bmatrix} \frac{7}{4} & \frac{5}{4} & \frac{1}{2} \\ 2 & 2 & 1 \\ \frac{-1}{4} & \frac{-1}{4} & 0 \end{bmatrix}$$

Questions for practice

1 If A= $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ Then show that $A^2 - 4A - 5I = 0$ and hence find A^{-1} **[CBSE 2015]**

2. If A= $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$, then find A^{-1} using elementary row operations.

3.Two schools P and Q want to award their selected students on the values of Discipline, Politeness and Punctuality. The school Pwants to award Rs. X each, Rs.Y each and RsZ each for the three respective values to its 3,2 and 1 students with a total award money of Rs. 1,000. School Q wants to spend Rs.1,500 to award its 4, 1 and 3 students on the respective values (by giving the same award money for the three values as before). If the total amount of awards for one prize on each value is Rs 600, using matrices, find the award money for each value. **Apart from the above three values, suggest one more value for awards**.

4.A typist charges Rs 145 for typing 10 English and 3 Hindi pages, while charges for typing 3 English and 10 Hindi pages are Rs 180. Using matrices , find the charges of typing 1 English and 1 Hindi page separately. However typist charged only Rs 2 per page from a poor student Shyam for 5 Hindi pages. How much less was charged from this poor boy? **Which values are reflected in this problem**