CHAPTER-5

CONTINUITY AND DIFFERENTIABILITY

Key concepts and formulae

OBJECTIVES: TO PROVIDE THE KNOWLEDGE OF CONTINUITY A ND DIFFERENTIATION OF REAL VALUED FUNCTIONS.

CONTENTS: 1. Meaning of Continuity

2. Continuity of real valued functions

3. Continuity at a point

4. continuity on the entire domain of the function

5. Points of discontinuity

6. To find the value(s) of constants when function is continuous at a point

7. Definition of derivative of a function(general and at a point) and Geometrical Meaning of Derivative.

8. Relationship between continuity and differentiation

9. Differentiation of polynomials

10. Product rule, Quotient rule and Chain rule of Differentiation

11. Differentiation of implicit and explicit functions

12 . Differentiation of rational functions

13. . Differentiation of functions in parametric forms

14 . Differentiation of Trigonometric functions

15. Differentiation of Inverse Trigonometric functions

16. . Differentiation of exponential and logarithmic functions

17. Logarithmic differentiation and higher derivatives

18. Rolle's and Mean Value Theorem

Methodology: Motivation through the examples from the environment, Demonstration, development of the concepts with the help of simple examples by involvement of the students, proceeding from simple to complex problems, Demonstration through Geometrical diagrams.

Important formulae of continuity and differentiability

1. f(x) is continuous at x=a if

$$\lim_{x\to a-} f(x) = \lim_{x\to a+} f(x) = f(a)$$

2.$\frac{d}{dx}(x^n) = nx^{n-1}$i.e$\frac{d}{dx}$(x)= 1.

3. $\frac{d}{dx}$(sinx)= cosx.

4. $\frac{d}{dx}$(cosx)= -sinx.

5.$\frac{d}{dx}$(tanx)=sec^2 x .

6. $\frac{d}{dx}$(cotx)= -$cosec^2$ x.

7.$\frac{d}{dx}$(sec x) = secxtanx.

8. $\frac{d}{dx}$(cosecx) = -cosecx cotx.

9.$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

10.$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

11.$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$

12.$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}.$

13.$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

14.$\frac{d}{dx}(cosec^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

15. $\frac{d}{dx}(e^x) = e^x$.

16(a).$\frac{d}{dx}$(log|x|) =$\frac{1}{x}$.

(b) $\frac{d}{dx}(\log_a x)$ =, $\frac{1}{x \log a}$ a > 0, a ≠ 1, x

17. $\frac{d}{dx}(a^x) = a^x \log a$.

18. **Product Rule For Derivative:**

$$\frac{d}{dx}[f(x).g(x)] = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)$$

19. **Quotient Rule For Derivative:**

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$$

20. **Chain Rule For Composite Functions:**

$$\frac{d}{dx}[fog(x)] = \frac{d}{dx}[f(g(x)] = \frac{d}{dx}f(g(x)) \times \frac{d}{dx}g(x)$$

21. Rolle's Theorem:

Let f be a real valued function on the closed interval $[a,b]$ s.t.

- It is continuous on [a,b]
- It is differentiable on (a,b)
- f (a) =f(b),then there exists at least a real number cε(a,b) such that f'(c)=0.
    - The geometrical meaning of Rolle's Theorem is that,if the conditions of Rolle's Theorem are satisfied,there exists at least one point (c,f(c)) on the curve where the tangent is parallel to x-axis.

22. **Mean Value Theorem/ Lagrange's Mean Value Theorem:**

Let f be a real valued function defined on a closed interval [a,b] s.t.

- It is continuous on [a,b]
- It is differentiable on (a,b)
- then there exists at least a real number cε(a,b) such that
- $f'(c) = \frac{f(b) - f(a)}{b-a}$.
- The geometrical meaning of Mean Value Theorem is that,if all the conditions of Mean Value Theorem are satisfied, then there exists at least one point (c,f(c)) on the curve where the tangent is parallel to the chord joining (a,f(a)) & (b,f(b)).

IMPORTANT BOARD QUESTIONS

One Mark Questions

1. Is sin x continuous at $x = \pi/2$ or not?
2. Differentiate y = sin $(x^2 + 5)$ w. r. t. x.
3. Differentiate sin $(\tan^{-1}e^x)$ w. r. t. x.
4. Differentiate $e^{\log x}$ w. r. t. x.
5. If f(1) =4 and $f^{/}(1)$ =2 ,find the value of the derivative of log(f(e^x)) at x=0
6. Find the derivative of $\sin^{-1} x$ w.r.t $\cos^{-1} x$

7. Verify the Rolle's theorem for the function $f(x) = x^2 - 2x + 1$ in the interval [-1,2]

Solutions

1 Ans. *Yes* , being a trigonometric function sin x is continuous in its domain.

2. Ans. $\frac{dy}{dx}$ =2xcos(x^2+5)

3. Ans. $e^x\cos(\tan^{-1}e^x)/1+e^{2x}$

4. Ans. 1

5.$\frac{1}{2}$

6. let u =$\sin^{-1} x$ and v= $\cos^{-1} x$

Then find du/dv = $\frac{\frac{du}{dx}}{\frac{dv}{dx}}$ = -1

7. Being polynomial function, It is continuous and differentiable in given interval f(-1)=4, f(2)=1

butf(-1)≠ f(2),Hence Roll's theorem is not satisfied

Four Marks Questions

1.If y =3e^{2x}+2e^{3x},then show that $y^{//} - 5y^{/} + 6y = 0$.

Solution. $y^{/}=6e^{2x}+6e^{3x}$,
$y^{//}=12e^{2x}+18e^{3x}$,
putting values of $y^{/}$ and $y^{//}$ in L.H.S of $y^{//} - 5y^{/} + 6y = 0$. And verify

2.If y = sin (log x), then prove that $x^2y'' + x\,y' + y = 0$.

. Solution

. $\frac{dy}{dx} = \frac{\cos(\log x)}{x}$.

Again differentiating w.r.t x

$y'' = \frac{-\cos(\log x)-\sin(\log x)}{x2}$.

$x^2y'' = \frac{-\cos(\log x)-\sin(\log x)}{} = -xy' - y$

Q3. . If $y = e^{ax}\,.\cos bx$, then prove that $\frac{d^2y}{dx^2} - 2a\,\frac{dy}{dx} + (a^2 + b^2)\,y = 0$

Solution.

$y = \boldsymbol{e^{ax}.\cos bx}$

$\frac{dy}{dx} = a\boldsymbol{e^{ax}.\cos bx} - b\boldsymbol{e^{ax}.\sin bx}$..........(1)

$\frac{dy}{dx} = ay - b\boldsymbol{e^{ax}.\sin bx}$

$\frac{d^2y}{dx^2} = a\frac{dy}{dx} - b(a\boldsymbol{e^{ax}.\sin bx} + b\boldsymbol{e^{ax}.\cos bx})$

$\frac{d^2y}{dx^2} = a\,\frac{dy}{dx} - (ay - \frac{dy}{dx}) - \boldsymbol{b^2}\,y$ (using 1)

$\frac{d^2y}{dx^2} - 2a\,\frac{dy}{dx} + (\boldsymbol{a^2} + \boldsymbol{b^2})y = 0$.

Hence Proved.

Q4. If $x = a\sin 2t\,(1 + \cos 2t)\ and\ y = b\cos 2t(1 - \cos 2t), then\ find\ \frac{dy}{dx}\ at\ t = \frac{\pi}{4}$

Solution.$x = a\sin 2t\ \ (1 + \cos 2t\ )$

$$y = b\cos 2t\,(1 - \cos 2t)$$

$$\frac{dx}{dt} = 2a\cos 2t\,(1 + \cos 2t) + a\sin 2t\ (-2\sin 2t\ )$$

$= 2a\cos 2t + 2a\,(\cos 2t)^2 + a\sin 2t\,(-2\sin 2t)^2$

$$= 2a\cos 2t + 2a\cos 4t$$

$$\frac{dy}{dt} = -2b\ \sin 2t\,(1-\cos 2t) + b\cos 2t\,(2\sin 2t\,)$$

$$= -2b\ \sin 2t + 2b\sin 2t\cos 2t + 2b\cos 2t\sin 2t$$

$$= -2b\sin 2t + 2b\sin 4t$$

$$\frac{dy}{dx} = \frac{-2b\sin 2t + 2b\sin 4t}{2a\cos 2t + 2a\cos 4t}$$

$$At\ t = \frac{\pi}{4}\ , \frac{dy}{dx} = \frac{b}{a}$$

Q5. Let f(x) = $\begin{cases} 1 \ \ ,if\ x \le 3 \\ ax+b\ ,\ if\ 3 < x < 5 \\ 7\ ,\ if\ \ x \ge 5 \end{cases}$. Find the values of a & b so that f(x) is a continuous function.

Solution. At x = 3

L.H.L = 1 , R.H.L = 3a + b and f(3) = 1

Since the function is continuous at x = 3

3 a + b = 1 ….(i)

At x = 5

L.H.L = 5 a + b , R.H.L = 7 and f(5) = 7

Since the function is continuous at x = 5

5 a + b = 7 ….(ii)

From (i) and (ii) , we obtain

a = 3 and b = - 8

Q6. Find dy/dx if y = $(\text{Cos } x)^x + (\text{Sin } x)^{1/x}$.

Solution.

inx)$^{1/x}$

fin let u = $(\text{Cos } x)^x$, v = $(\text{Sin } x)^{1/x}$

y = u + v

dy/dx = du/dv + dv/dx

u = (Cos x)^{x}

log u = x log.cosx

finding du/dx = (Cos x)^{x} [-x tan x + log(cosx)

Similarly from v = (S ding dv/dx = (Sin x) $^{1/x}$ [x cotx + log(Sinx)]

HOTS

1. Find $\frac{dy}{dx}$, if $y^x + x^y + x^x = a^b$.

Solution.

Let v =x^y

$\log v$ = $y \log x \; x^x + x^y + y^x = a^b$

Let u = x^x

$\log u$ = $x \log x$

$\frac{1}{u}\frac{du}{dx}$ = x.$\frac{1}{x}$ + $\log x$

$\frac{du}{dx}$ = x^x (1+ $\log x$)

$\frac{1}{v}\frac{dv}{dx}$ = ($\frac{y}{x}$ + $\log x$ $\frac{dy}{dx}$)

$\frac{dv}{dx}$ = x^y ($\frac{y}{x}$ +$\log x$ $\frac{dy}{dx}$)

Let w = y^x

$\log w$ = $x \log y$

$\frac{1}{w}.\frac{dw}{dx} = (\frac{x}{y}\frac{dy}{dx} + \log y)$

$\frac{dw}{dx}$ = y^x ($\frac{x}{y}\frac{dy}{dx}$ + $\log y$)

(1) Can be written as

u +v+ w = a^b

$$\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}=0$$

x^x (1+ $\log x$) + x^y ($\frac{y}{x}$ +$\log x$ $\frac{dy}{dx}$) + y^x ($\frac{x}{y}\frac{dy}{dx}$ + $\log y$) = 0

$\frac{dy}{dx}$($x^y \log x + y^x.\frac{x}{y}$) = $x^x + x^x \log x\, x^y \frac{y}{x} + y^x \log y \frac{dy}{dx}$

$$\frac{dy}{dx}=\frac{x^x+\ x^x\log x+yx^{y-1}+y^x.\log y}{x^y.\log x+\ xy^{x-1}}$$

Q2. Show that function f(x) defined by

$$f(x)=\begin{cases}\frac{sinx}{x}+cosx, & x>0\\ 2, & x=0,\\ \frac{4(1-\sqrt{1-x})}{x}, & x<0\end{cases}$$ **is continuous at x=0**

Solution.

Solving LHL = 2.

Solving RHL = 2

Also f(0) = 2

LHL = RHL = f(0)

f(x) is continuous at x = 2.

Q3. If y = $\tan^{-1}\left(\frac{5ax}{a^2-6x^2}\right)$ **then show that** $\frac{dy}{dx}=\frac{3a}{a^2+9x^2}+\frac{2a}{a^2+4x^2}$

Solution.

$$y=\tan^{-1}\left(\frac{5ax}{a^2-6x^2}\right)=\tan^{-1}\left[\frac{\frac{5ax}{a^2}}{\frac{a^2-6x^2}{a^2}}\right]$$

$$=\tan^{-1}\left[\frac{\frac{3x}{a}+\frac{2x}{a}}{1-\frac{3x}{a}\times\frac{2x}{a}}\right]=\tan^{-1}\frac{3x}{a}+\tan^{-1}\frac{2x}{a}$$

$$hence,\frac{dy}{dx}=\frac{1}{1+\left(\frac{3x}{a}\right)^2}\times\frac{3}{a}+\frac{1}{1+\left(\frac{2x}{a}\right)^2}\times\frac{2}{a}$$

$$=\frac{3a}{a^2+9x^2}+\frac{2a}{a^2+4x^2}$$

Q4 If $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, prove that $\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$.

Sol : Put x = sin α, y = sin β

Cosα + cos β = a (sin α – sin β)

$\cot\left(\frac{\alpha-\beta}{2}\right)$ = a $\Rightarrow$ $\sin^{-1}x-\sin^{-1}y=2\cot^{-1}a$

$\Rightarrow \frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$

Q5.Find $\frac{dy}{dx}$ when $y=x^{\log x}+(\log x)^x$

Solution.

Take y=u+v where $u(x)=x^{\log x}$ and $v(x)=(\log x)^x$ so that $\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$

$u(x)=x^{\log x}\Rightarrow \log u=\log x\log x$ => $\frac{du}{dx}=x^{\log x}\left(\frac{2}{x}\log x\right)$

$v(x)=(\log x)^x\Rightarrow \log v=x\log(\log x)$ => $\frac{dv}{dx}=(\log g)^x\left[\frac{1}{\log x}+\log(\log x)\right]$

$\frac{dy}{dx}=x^{\log x}\left(\frac{2}{x}\log x\right)+(\log g)^x\left[\frac{1}{\log x}+\log(\log x)\right]$

Q6. Differentiate $\tan^{-1}\left[\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right]$ with respect to x

Solution.Let $y = \tan^{-1}\left[\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right] = \tan^{-1}\left[\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right] =$ (by substitution $x^2 = \cos\theta$)

$\tan^{-1}\left[\frac{\sqrt{2\cos^2\frac{\theta}{2}}-\sqrt{2\sin^2\frac{\theta}{2}}}{\sqrt{2\cos^2\frac{\theta}{2}}+\sqrt{2\sin^2\frac{\theta}{2}}}\right] = \tan^{-1}\left[\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right] = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\} = \frac{\pi}{4}-\frac{\theta}{2} = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x^2$

$\frac{dy}{dx} = \frac{x}{\sqrt{1-x^4}}$

Q7.If y = $[\log(x+\sqrt{x^2+1})]^2$, *Show that* $(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=2$

Solution.

y = $[\log(x+\sqrt{x^2+1})]^2$

$\frac{dy}{dx} = 2\,[\log(x+\sqrt{x^2+1})]\frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{1}{2\sqrt{x^2+1}}2x\right)$

$\frac{dy}{dx} = 2\,[\log(x+\sqrt{x^2+1})]\frac{1}{\sqrt{x^2+1}}$

$(x^2+1)\frac{dy}{dx} = 4\,[\log(x+\sqrt{x^2+1})]^2$ after squaring $(x^2+1)\left(\frac{dy}{dx}\right)^2 = 4y$

diff. both sides $(x^2+1)2\frac{dy}{dx}\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 2x = 4\frac{dy}{dx}$ => $(x^2+1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=2$

<u>MORE QUESTIONS FOR PRACTICE</u>

1.Prove that $\frac{d^2y}{dx^2}$+y =0 ,if y= 5cosx -3sinx

2. Find *k* for which the function *f(x)* = $\begin{cases}\frac{\sin 5x}{3x} & x\neq 0\\ k & x=0\end{cases}$ is continuous at *x* = 0

3.Find$k$, if $f(x)= \frac{x^2 - 16}{x - 4}$, $x \neq 4$ and $f(x) = k$ if $x = 4$, is continuous at $x = 4$.

4.Discuss the continuity of the function $f(x) = x^3 + x^2 – 1$

5. If y= 3cos(logx)+4sin(logx), show that $x^2 \frac{d^2y}{dx^2}$ +x$\frac{dy}{dx}$ +y =0 .

6.If x$\sqrt{1+y}$ + y$\sqrt{1+x}$ = 0, show that $\frac{dy}{dx}$ = - $((1+x)^{-2}$

7.If$y = \sqrt{x + \sqrt{x + \sqrt{x + ..\infty}}}$, prove that $\frac{dy}{dx} = \frac{1}{2y-1}$

8. If x = $\sqrt{a^{\sin^{-1}(t)}}$, **y =** $\sqrt{a^{\cos^{-1}(t)}}$ **, show that** $\frac{dy}{dx} = -\frac{Y}{X}$

9. If y=$\sqrt{x}^{\sqrt{x}^{\sqrt{x}^{\sqrt{x}......to\infty}}}$,prove that x$\frac{dy}{dx} = \frac{y^2}{2-ylogx}$.

10.If y=e^{ax}cosbx , then prove that $\frac{d^2y}{dx^2}$ -2a $\frac{dy}{dx}$ (a^2+b^2)y=0 .

11. If x=asin2t(1+cos2t) and y= bcos2t (1-cos2t) then find $\frac{dy}{dx}$ at t =$\frac{\pi}{4}$.

12. Find the value of ‘k’ so that the function f is continuous at $x = \frac{\pi}{2}$

$$f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}; & x \neq \frac{\pi}{2} \\ 3; & x = \frac{\pi}{2} \end{cases}$$

13. If y =$(x + \frac{1}{x})^x + x^{(1+1/x)}$, then find dy/dx