KEY CONCEPTS/IMPORTANT FORMULAE

VECTORS

* Position vector of point A (x, y, z) = $\vec{OA} = x\hat{i} + y\hat{j} + z\hat{k}$

* If $A(x_1, y_1, z_1)$ and point $B(x_2, y_2, z_2)$ then $\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

* If $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$; $|\vec{a}| = \sqrt{x^2 + y^2 + z^2}$

* Unit vector parallel to $\vec{a} = \frac{\vec{a}}{|\vec{a}|}$

* Scalar Product (dot product) between two vectors : $\vec{a}.\vec{b} = |\vec{a}||\vec{b}| \cos\theta$; θ is angle between the vectors

* $\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

* If $\vec{a} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ and $\vec{b} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ then $\vec{a}.\vec{b} = a_1a_2 + b_1b_2 + c_1c_2$

* If $\vec{a}$ is perpendicular to $\vec{b}$ then $\vec{a}.\vec{b} = 0$

* $\vec{a}.\vec{a} = |\vec{a}|^2$

* Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$ and Projection of $\vec{b}$ on $\vec{a} = \frac{\vec{a}.\vec{b}}{|\vec{a}|}$

* Vector product between two vectors :

  $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$; $\hat{n}$ is the normal unit vector which is perpendicular to both $\vec{a}$ & $\vec{b}$

* $\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

* If $\vec{a}$ is parallel to $\vec{b}$ then $\vec{a} \times \vec{b} = 0$

  * Area of triangle (whose sides are given by $\vec{a}$ and $\vec{b}$) = $\frac{1}{2}|\vec{a} \times \vec{b}|$

  * Area of parallelogram (whose adjacent sides are given by $\vec{a}$ and $\vec{b}$) = $|\vec{a} \times \vec{b}|$

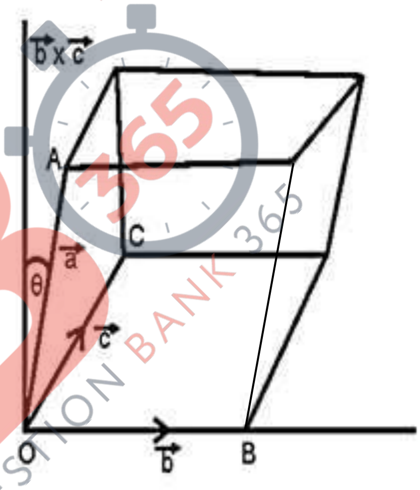
  * Area of parallelogram (whose diagonals are given by $\vec{a}$ and $\vec{b}$) = $\frac{1}{2}|\vec{a} \times \vec{b}|$

I. SCALAR TRIPLE PRODUCT

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors. Then the scalar $(\vec{a} \times \vec{b}).\vec{c}$ is called the scalar triple product of $\vec{a}$, $\vec{b}$ and $\vec{c}$ and is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$

$\therefore [\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \times \vec{b}).\vec{c}$

II. GEOMETRICAL INTERPRETATION OF A SCALAR TRIPLE PRODUCT



If three co-terminus edges OA, OB and OC of a parallelopiped are represented by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively, then $\vec{b} \times \vec{c}$ represents the vector area of the base of the paralleopiped and the height of the parallelopiped is the projection of $\vec{a}$ along the normal to the plane containing Vectors $\vec{b}$ and $\vec{c}$, i.e., along $\vec{b} \times \vec{c}$

Magnitude of this projection $= \dfrac{\vec{a}.(\vec{b} \times \vec{c})}{|\vec{b} \times \vec{c}|}$

$\therefore$ Volume of the parallelopiped = (Area of base) x (Height)

$$= \frac{|\vec{b} \times \vec{c}|\ |\vec{a}.(\vec{b} \times \vec{c})|}{|\vec{b} \times \vec{c}|} = |\vec{a}.(\vec{b} \times \vec{c})|$$

{ Modulus has been taken as area is always positive}

Thus, if $\vec{a}$, $\vec{b}$ and $\vec{c}$ represent the three co-terminus edges of a parallelgram then its volume =

$$\vec{a}.(\vec{b} \times \vec{c}) \text{ or } [\vec{a}\ \vec{b}\ \vec{c}]$$

III. SCALAR TRIPLE PRODUCT IN TERMS OF RECTANGULAR COMPONENTS

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

then $\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (b_2 c_3 - b_3 c_2)\hat{i} - (b_1 c_3 - b_3 c_1)\hat{j} + (b_1 c_2 - b_2 c_1)\hat{k}$

$$\vec{a}.(\vec{b} \times \vec{c}) = a_1(b_2 c_3 - b_3 c_2) - a_2(b_1 c_3 - b_3 c_1) + a_3(b_1 c_2 - b_2 c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Remarks : If for any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, $[\vec{a}\ \vec{b}\ \vec{c}] = 0$, then the volume of the parallelopipped with the three co-terminus edges $\vec{a}$, $\vec{b}$ and $\vec{c}$, is zero, which is possible only if $\vec{a}$, $\vec{b}$ and $\vec{c}$, are co-planar vectors.

Thus, $[\vec{a}\ \vec{b}\ \vec{c}] = 0 \Leftrightarrow \vec{a}, \vec{b}$ and $\vec{c}$ are co-planar

IV. PROPERTIES OF SCALAR TRIPLE PRODUCT

1. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three vectors, then

$$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$$

Proof : Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{bmatrix} \vec{b} & \vec{c} & \vec{a} \end{bmatrix}$$

$$\therefore \begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = \begin{bmatrix} \vec{b} & \vec{c} & \vec{a} \end{bmatrix} \_\_\_(i)$$

Similarly, it can be vertified that $\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = \begin{bmatrix} \vec{c} & \vec{a} & \vec{b} \end{bmatrix} \_\_\_(ii)$

from (i) and (ii), we see that

$$\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = \begin{bmatrix} \vec{b} & \vec{c} & \vec{a} \end{bmatrix} = \begin{bmatrix} \vec{c} & \vec{a} & \vec{b} \end{bmatrix}$$

$\Rightarrow$ If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are cyclically permuted, the value of the scalar Triple Product remains unaltered.

2. In scalar triple product, the position of dot and cross can be interchanged, provided the cyclic order of vectors remains the same.

Proof: Since $\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = \begin{bmatrix} \vec{c} & \vec{a} & \vec{b} \end{bmatrix}$

$$\Rightarrow \vec{a}.\left(\vec{b} \times \vec{c}\right) = \vec{c}.\left(\vec{a} \times \vec{b}\right)$$
$$\text{or } \vec{a}.\left(\vec{b} \times \vec{c}\right) = \vec{c}.\left(\vec{a} \times \vec{b}\right) = \left(\vec{a} \times \vec{b}\right).\vec{c}$$

3. The value of the scalar triple product remains the same in magnitude, but changes the sign, if the cyclic order of $\vec{a}$, $\vec{b}$ and $\vec{c}$ is changed.

Proof: $\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = \vec{a}.\left(\vec{b} \times \vec{c}\right) = -\vec{a}.\left(-\vec{c} \times \vec{b}\right) = \vec{a}\left(\vec{c} \times \vec{b}\right) = -\begin{bmatrix} \vec{a} & \vec{c} & \vec{b} \end{bmatrix}$

4. The scalar triple product of three vectors is zero if any two of the given vectors are equal.

Proof: Let $\vec{a} = \vec{b}$

$$\therefore \begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = \begin{bmatrix} \vec{a} & \vec{a} & \vec{c} \end{bmatrix} = \left(\vec{a} \times \vec{a}\right).\vec{c} = 0$$

Similarly, if $\vec{b} = \vec{c}$ or $\vec{c} = \vec{a}$, $\begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = 0$

5. For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ and scalar λ, we have

$$\therefore \left[\lambda\vec{a} \quad \vec{b} \quad \vec{c}\right] = \lambda\left[\vec{a} \quad \vec{b} \quad \vec{c}\right]$$

Proof: $\left[\lambda\vec{a} \quad \vec{b} \quad \vec{c}\right] = \left(\lambda\vec{a} \times \vec{b}\right).\vec{c}$

$$= \lambda\left(\vec{a} \times \vec{b}\right).\vec{c} = \lambda\left[\vec{a} \quad \vec{b} \quad \vec{c}\right]$$

6. The scalar triple product of three vector is zero if any two of them are parallel or collinear

Proof: let $\vec{a}$ be parallel (or collinear) to $\vec{b}$

$\therefore \vec{a} = \lambda\vec{b}$ for some scalar λ

$$\therefore \left[\vec{a} \quad \vec{b} \quad \vec{c}\right] = \left[\lambda\vec{b} \quad \vec{b} \quad \vec{c}\right] = \lambda\left[\vec{b} \quad \vec{b} \quad \vec{c}\right] = \lambda.0 = 0$$

Let us now take some examples:

Example1: If $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{c} = 3\hat{i} + 4\hat{j} - \hat{k}$

then find $\vec{a}.\left(\vec{b} \times \vec{c}\right)$ and $\left(\vec{a} \times \vec{b}\right).\vec{c}$. Is $\vec{a}.\left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \times \vec{b}\right).\vec{c}$?

Solution:

$$\vec{a}.\left(\vec{b} \times \vec{c}\right) = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -3 \\ 3 & 4 & -1 \end{vmatrix} = 2\,(-2+12) +3\,(-1+9) +4\,(4-6) = 20 + 24 - 8 = 36$$

$$\left(\vec{a} \times \vec{b}\right).\vec{c} = \vec{c}.\left(\vec{a} \times \vec{b}\right) = \begin{vmatrix} 3 & 4 & -1 \\ 2 & -3 & 4 \\ 1 & 2 & -3 \end{vmatrix} = 3\,(9-8) -4\,(-6-4) -1\,(4+8(= 3 + 40 - 7 = 36$$

$$\therefore \vec{a}.\left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \times \vec{b}\right).\vec{c}$$

3-D GEOMETRY

** STRAIGHT LINE:

* Equation of line(one point form)

(Caretesian Form) – Equation of line passing through a point (x_1, y_1, z_1) with direction cosines a, b, c:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

(Vector form)

Equation of line passing through a point $\vec{a}$ and in the direction of $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$

* Equation of line(Two point form)

(Caretesian Form)

Equation of line passing through two point (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$

(Vector form)

Equation of line passing through two points $\vec{a}$ & $\vec{b}$ and in the direction of $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\left(\vec{b} - \vec{a}\right)$

* Equation of line passing through a point (x_1, y_1, z_1) and parallel to the line : $\frac{x - \alpha}{a} = \frac{y - \beta}{b} = \frac{z - \gamma}{c}$ is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

* Shortest distance between two skew lines : if lines are $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$, $\vec{r} = \vec{a_2} + \lambda\vec{b_2}$

then Shortest distance $= \frac{\left(\vec{a_2} - \vec{a_1}\right).\left(\vec{b_1} \times \vec{b_2}\right)}{\left|\vec{b_1} \times \vec{b_2}\right|}$ $\vec{b_1} \times \vec{b_2} \neq 0$

If lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ $\vec{r} = \vec{a_2} + \lambda\vec{b_2}$ *are Parallel then*

Shortest distance $= \frac{\left|\left(\vec{a_2} - \vec{a_1}\right) \times \vec{b_1}\right|}{\left|\vec{b_1}\right|}$ $\vec{b_1} \times \vec{b_2} = 0$

** Direction Cosines and Direction Ratios :

If a line makes angles α, β and γ with x, y and z axes respectively then cos α, cos β and cos γ are the direction cosines denoted by l, m and n respectively and $l^2 + m^2 + n^2 = 1$

Any three numbers proportional to direction cosines are called direction ratios denoted by a, b, c

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} \qquad l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}, \quad n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}},$$

** **PLANE** :

* Equation of plane is $ax + by + cz + d = 0$ where a, b & c are direction ratios of normal to the plane

* Equation of plane passing through a point (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ where a, b & c are direction ratios of normal to the plane

* Equation of plane in intercept form is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, where a, b, c are intercepts on the axes

* Equation of plane in normal form $lx + my + nz = p$ where l, m, n are direction cosines of normal to the plane and p is length of perpendicular from origin to the plane .

* Equation of plane passing through three points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3)

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

* Equation of plane passing through two points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and perpendicular to the plane $ax + by + cz + d = 0$ or parallel to the line $\frac{x - \alpha_1}{a} = \frac{y - \beta_1}{b} = \frac{z - \gamma_1}{c}$ is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a & b & c \end{vmatrix} = 0$

* Equation of plane passing through the point (x_1, y_1, z_1) and perpendicular to the planes $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$ or parallel to the lines $\frac{x - \alpha_1}{a_1} = \frac{y - \beta_1}{b_1} = \frac{z - \gamma_1}{c_1}$ and $\frac{x - \alpha_2}{a_2} = \frac{y - \beta_2}{b_2} = \frac{z - \gamma_2}{c_2}$ is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

* Equation of plane contaning the line $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and passing through the point (x_2, y_2, z_2) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0$$

* **Conditon for coplanarlines:**

$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ are coplanar if

$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$ and equation of common plane is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

* Equation of plane passing through the intersection of two planes $a_1x + b_1y + c_1z = 0$, $a_2x + b_2y + c_2z = 0$ is $(a_1x + b_1y + c_1z) + \lambda(a_2x + b_2y + c_2z) = 0$

* Perpendicular distance from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is $\left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$

* Distance between two parallel planes $ax + by + cz + d_1 = 0$, $ax + by + cz + d_2 = 0$ is $\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$

(A)IMPORTANT BOARD QUESTIONS

On Vectors

*1. Find the projection of $\vec{a} = (4\vec{i} - 3\vec{j} + \vec{k})$ on $\vec{b} = 2\hat{i} - 3\hat{j} + 4\hat{k}$. (CBSE 2010)

Solution : $\vec{a} = (4\vec{i} - 3\vec{j} + \vec{k})$, $\vec{b} = 2\hat{i} - 3\hat{j} + 4\hat{k}$

Projection of $\vec{a}$ on $\vec{b}$ $= \frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{21}{\sqrt{29}}$

2.Write a vector of magnitude 15 units in the direction of vector $\hat{i} - 2\hat{j} + 2\hat{k}$. (CBSE 2010)

Solution: Let $\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$

$$\left|\vec{a}\right| = \sqrt{1 + 4 + 4} = 3$$

$$\hat{a} = \frac{\vec{a}}{\left|\vec{a}\right|} = \frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k}) \qquad \frac{1}{2}$$

Vector of magnitude 15 units in the direction of vector $\vec{a} = 15\hat{a} = 5\hat{i} - 10\hat{j} + 10\hat{k}$ $\frac{1}{2}$

3. What is the cosine of the angle which the vector $\sqrt{2}\hat{i} + \hat{j} + \hat{k}$ makes with y – axis ? (CBSE 2010)

Solution: D.rs of the vector $\sqrt{2}\hat{i} + \hat{j} + \hat{k}$ are $< \sqrt{2}, 1, 1 >$

D.rs of the y –axis are $<0, 1, 0>$ $\frac{1}{2}$

Let θ be the angle between the given vector and y – axis

$$\therefore \cos\theta = \frac{\left|\sqrt{2}(0) + 1(1) + 1(0)\right|}{\sqrt{4 + 1 + 1}\sqrt{0 + 1 + 0}} = ½ \qquad \frac{1}{2}$$

4. If $\vec{a} = x\hat{i} + 2\hat{j} - z\hat{k}$ and $\vec{b} = 3\hat{i} - y\hat{j} + \hat{k}$ are two equal vectors, then write the value of x + y + z. (CBSE 2013)

Solution: $x + y + z = 0$

5.Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $(2\vec{a}+\vec{b})$ and $(\vec{a}-3\vec{b})$ respectively, externally in the ratio 1 : 2. Also, show that P is the mid-point of the line segment RQ **(CBSE 2010)**

Solution: Let $\vec{OP}=2\vec{a}+\vec{b}$ and $\vec{OQ}=\vec{a}-3\vec{b}$ $\frac{1}{2}$

Ratio 1:2 externally

$$\therefore \vec{OR}=\frac{1(\vec{a}-3\vec{b})-2(2\vec{a}+\vec{b})}{1-2}=3\vec{a}+5\vec{b} \qquad 1\frac{1}{2}$$

Consider the mid point of RQ as P = $\frac{\vec{OR}+\vec{OQ}}{2}=\frac{4\vec{a}+2\vec{b}}{2}=2\vec{a}+\vec{b}=\vec{OP}$ 2

6. Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x}-\vec{a}).(\vec{x}+\vec{a})=15$ **(CBSE 2010)**

Solution: Here $\vec{a}$ is a unit vector $\therefore \left|\vec{a}\right|=1$

$$(\vec{x}-\vec{a}).(\vec{x}+\vec{a})=15 \qquad \frac{1}{2}$$

$$\left|\vec{x}\right|^2-\left|\vec{a}\right|^2=15$$

$$\left|\vec{x}\right|=4$$

7.The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with the unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ. **[CBSE 2014]**

Solution: Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$; $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$; $\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$

$\vec{b}+\vec{c}=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}=\vec{d}$ (say)

$$\hat{d}=\frac{\vec{d}}{|\vec{d}|}$$

According to question

$\vec{a}.\hat{d}=1$

$\rightarrow \lambda=1$

8. Show that the points with position vectors $4\hat{i}+8\hat{j}+12\hat{k}$, $2\hat{i}+4\hat{j}+6\hat{k}$, $3\hat{i}+5\hat{j}+4\hat{k}$ and $5\hat{i}+8\hat{j}+5\hat{k}$ are coplanar. **[CBSE 2015]**

Solution : The given vectors are coplanar $\Rightarrow \begin{bmatrix} \vec{AB} & \vec{AC} & \vec{AD} \end{bmatrix} = 0$

$$\Rightarrow \begin{vmatrix} -2 & -4 & -6 \\ -1 & -3 & -8 \\ 1 & 0 & -7 \end{vmatrix} = 0$$

Which gives 0=0

Hence proved

On 3 – D Geometry

1. Find the value of λ if $\vec{a} = \hat{i} + 3\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} - \hat{k}$ and $\vec{c} = \lambda\hat{j} + 3\hat{k}$ are coplanar.

(CBSE 2011)

Solution: The given vectors are coplanar $\Rightarrow \begin{bmatrix} \vec{a} & \vec{b} & \vec{c} \end{bmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & -1 \\ 0 & \lambda & 3 \end{vmatrix} = 0$$

$\Rightarrow$ 1(-3+λ) – 2(9 - λ) =0

$\Rightarrow$ $\lambda = 7$

2. Show that the four points (0, –1, –1), (4, 5, 1), (3, 9, 4) and (– 4, 4, 4) are coplanar

Solution: The given vectors are coplanar $\Rightarrow \begin{bmatrix} \vec{AB} & \vec{AC} & \vec{AD} \end{bmatrix} = 0$ (CBSE 2015)

$$\Rightarrow \begin{vmatrix} 4 & 6 & 2 \\ 3 & 10 & 5 \\ -4 & 5 & 5 \end{vmatrix} = 0$$

Which gives 0=0

Hence proved

3. Find the shortest distance between the lines whose vector equations are

$\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$ (CBSE 2010) Solution:

$\overrightarrow{(a_2 - a_1)} = (\hat{i} - \hat{k})$

$\overrightarrow{b_1} \times \overrightarrow{b_2} = (3\hat{i} - \hat{j} - 7\hat{k})$

$\left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right| = \sqrt{59}$

$$sd = \left|\frac{\overrightarrow{(a_2 - a_1)}.\overrightarrow{b_1} \times \overrightarrow{b_2}}{\left|\overrightarrow{b_1} \times \overrightarrow{b_2}\right|}\right|$$

$$= \left|\frac{(\hat{i} - \hat{k})(3\hat{i} - \hat{j} - 7\hat{k})}{\sqrt{59}}\right|$$

$$= \frac{10}{\sqrt{59}}$$

4. Find the angle between the lines

$\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$ and

$\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$ (CBSE 2011)

Solution:

$\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$, $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$ and 1

$$\cos\theta = \left|\frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

$$= \left|\frac{19}{21}\right|$$

$$\theta = \cos^{-1}\left(\frac{19}{21}\right)$$

5. Find a unit vector perpendicular to both of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ where $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ (CBSE 2011)

Solution:

$\vec{a} + \vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{a} - \vec{b} = -\hat{j} - 2\hat{k}$

$\vec{c} = 2\hat{i} + 2\hat{j} + 2\hat{k}$

$$\hat{c} = \frac{\vec{c}}{|\vec{c}|} = \pm\frac{1}{2\sqrt{2}}(2\hat{i} + 2\hat{j} + 2\hat{k})$$

6. Find the distance of a point $(2, 5, -3)$ from the plane $\vec{r} \cdot (6\hat{i} - 3\hat{j} + 2\hat{k}) = 4$

Solution: Eq of plane in Cartesian form is 6x-3y+2z-4=0

Its distance from point $(2,5,-3)$

$$D=\left|\frac{6\times 2-3\times 5+2\times(-3)-4}{\sqrt{36+9+4}}\right|=13/7 \text{ units}$$

7. Find the distance of the point P(–1, –5, –10) from the point of intersection of the line joining the points A(2, –1, 2) and B(5, 3, 4) with the plane x – y + z = 5. (CBSE 2014)

Solution Eqn of line $=\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{2}=k$

aribitary point $p(3k+2, 4k-1, 2k+2)$

put p in eqn of plane

$3k+2-4k+1+2k+2=5$

$k=0$, hence $p(2,-1,2)$

$d=\sqrt{(-1-2)^2+(-5+1)^2+(-10-2)^2}=13$

8. Find the coordinates of the foot of the perpendicular and the perpendicular distance of the point (1,3,4) from the plane 2x – y + z + 3 = 0. Find also, the image of the point in the plane.

Solution: Foot of perpendicular (-1,4,3), Image (-3,5,2), Distance $=\sqrt{6}$ units

9. Find the equation of the plane which contains the line of intersection of the planes $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})-4=0$, $\vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})+5=0$ and which is perpendicular to the plane $\vec{r}\cdot(5\hat{i}+3\hat{j}-6\hat{k})+8=0$. (CBSE 2011)

Solution:

Req eqn : $\vec{r}\cdot(\vec{n_1}+p\vec{n_2})=d_1+pd_2$

$\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k}+p(2\hat{i}+\hat{j}-\hat{k}))=4-5p$

$\vec{r}\cdot((1+2p)\hat{i}+(2+p)\hat{j}+(3-p)\hat{k})=4-5p \quad *(1)$

its perpendicular to $(5\hat{i}+3\hat{j}-6\hat{k})+8$

$\Rightarrow (1+2p).5+(2+p).3+(3-p).-6=0$

$p=\frac{7}{19}$, put in eqn (1)

$\vec{r}\cdot((1+2\times 7/19)\hat{i}+(2+7/19)\hat{j}+(3-7/19)\hat{k})=4-5\times 7/19$

$\vec{r}\cdot((33/19)\hat{i}+(45/19)\hat{j}+(50/19)\hat{k})=41/19$

$\vec{r}\cdot((33)\hat{i}+(45)\hat{j}+(50)\hat{k})=41$

10. If lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect , then find the value of k and hence find the equation of the plane containing these lines.(CBSE 2015)

Sol. Let given lines are L_1 : $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$

And L_2 : $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$

Any point on line L_1 is ($2\lambda+1$, $3\lambda-1$, $4\lambda+1$)

Any point on line L_2 is($\mu+3$, $2\mu+k$, μ) 1

The lines will intersect if these points coincides

If $2\lambda+1=\mu+3$

$3\lambda-1=2\mu+k$

$4\lambda+1=\mu$

Taking first two $2\lambda-\mu=2$ ----------(1)

Taking middle two $3\lambda-\mu=k+1$ ----------(2)

Taking last two $4\lambda-\mu=-1$ ----------(3)

Solving 1 and 3 we get $\lambda=-3/2$, $\mu=-5$

Putting in eqn 6 we get k = 9/2 1

The equation of plane containing given lines is

$$\begin{vmatrix} x-1 & y+1 & z-1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix}=0$$ 1

$\Rightarrow (x-1)(3-8)-(y+1)(2-4)+(z-1)(4-3)=0$

$5x-2y-z-6=0$

1

(B) IMPORTANT QUESTIONS

Q1. Show that the straight lines whose direction cosines are given by 2 l + 2 m – n = 0 and m n + n l + l m = 0 are at right angle .

Q2. If $l_1, m_1, n_1 ; l_2, m_2, n_2$ and l_3, m_3, n_3 are the direction cosines of three mutually perpendicular lines , prove that the line whose direction cosines are proportional to $l_1 + l_2 + l_3 ; m_1 + m_2 + m_3$ and $n_1 + n_2 + n_3$ makes equal angles with them.

Q3. A line passing through the point A with position vector $\vec{a} = 4\hat{i} + 2\hat{j} + 2\hat{k}$ is parallel to the vector $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$. Find the length of the perpendicular drawn on this line from a point P with position vector $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k}$.

Q4. Find the equation of the plane through the intersection of the planes $\vec{r}.(\hat{i} + 3\hat{j}) - 6 = 0$ and $\vec{r}.(3\hat{i} - \hat{j} - 4\hat{k}) = 0$, whose perpendicular distance from the origin is unity .

Q5. A vector $\vec{n}$ of magnitude 8 units is inclined to x-axis at 45^0 , y-axis at 60^0 and an acute angle with z-axis . If a plane passes through a point $(\sqrt{2}, -1, 1)$ and is normal to $\vec{n}$. Find its equation in vector form .

Q6. Find the equations of the two lines through the origin which intersect the line $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$ at angles of $\frac{\pi}{3}$ each.

Q7. Find the volume of the parallelopiped whose coterminus edges are represented by vectors $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$.

Q8. Two systems of rectangular axis have the same origin . If a plane cuts them at distances a , b ,c and a', b', c' respectively from the origin , prove that : $a^{-2} + b^{-2} + c^{-2} = (a')^{-2} + (b')^{-2} + (c')^{-2}$.

Q9. Find the distance of the point (1 , - 2 , 3) from the plane x – y + z = 5 , measured parallel to the line $\frac{x-1}{2} = \frac{y-3}{3} = \frac{z+2}{-6}$.

Q10. A plane meets t e coordinates axes in A , B ,C such that the centroid of the triangle ABC is the point (α, β, γ). Show that the equation of the plane is $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3$.

SOLUTIONS

Q3. The equation of the line passing through the point A and parallel to $\vec{b}$, in cartesian form is :

$$\frac{x-4}{2} = \frac{y-2}{3} = \frac{z-2}{6}$$

Let Q (α, β, γ) be the foot of the perpendicular drawn from point P to the above line.

Coordinate of the point P are P(1, 2 ,3).

Since Q lies on the above line,therefore

$$\frac{\alpha-4}{2}=\frac{\beta-2}{3}=\frac{\gamma-2}{6}=\lambda$$

$$\Rightarrow \alpha=2\lambda+4,\ \beta=3\lambda+2,\ \gamma=6\lambda+2$$

Since PQ is perpendicular to given line therefore

$$2\alpha+3\beta+6\gamma-26=0$$

$$\Rightarrow 2(2\lambda+4)+3(3\lambda+2)+6(6\lambda+2)-26=0$$

$$\lambda=0$$

Thus the coordinates of Q are Q (4 , 3 , 2)

Therefore the length of the perpendicular is :

$$|PQ|=\sqrt{(4-1)^2+(2-2)^2+(2-3)^2}$$
$$=\sqrt{9+0+1}=\sqrt{10}\ units$$

Q4. Required equation of the plane are:

2 x + y – 2 z – 3 = 0 and x – 2y – 2z + 3 = 0.

Q5. Required equation of the plane is $\vec{r}\cdot(\sqrt{2}\,\hat{i}+\hat{j}+\hat{k})=2$.

Q6. Given equation of the line is $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}=\lambda$

Any point on the above line is $P(2\lambda+3,\ \lambda+3,\ \lambda)$

Now direction ratio of OP is $<2\lambda+3-0,\ \lambda+3-0,\ \lambda-0>$
$<2\lambda+3,\ \lambda+3,\ \lambda>$

As Op makes an angle of $\frac{\pi}{3}$ with the given line

$$\cos\frac{\pi}{3}=\frac{|2(2\lambda+3)+1(\lambda+3)+1\times\lambda|}{\sqrt{(2\lambda+3)^2+(\lambda+3)^2+\lambda^2}}$$

$$\frac{1}{2}=\frac{4\lambda+6+\lambda+3+\lambda}{\sqrt{6\lambda^2+18\lambda+18}\sqrt{6}}$$

$$\lambda=-1\ or\ \lambda=-2$$

Coordinates of the point P are:

P (1 , 2, - 1) and P (- 1 , 1 , - 2)

Hence the required equation of the lines are : $\frac{x}{1}=\frac{y}{2}=\frac{z}{-1}$ and $\frac{x}{-1}=\frac{y}{1}=\frac{z}{-2}$.

Q7. Here $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \\ 3 & -1 & 2 \end{vmatrix}$

$= 2 (4 + 30) - 1 (2 – 9) + (-1) (- 1 – 6)$

$= 14 + 7 + 7 = 28$

Hence the required volume of the parallelopiped is $[\vec{a}\ \vec{b}\ \vec{c}]$ = 28 cubic units.

Q9. Let P(1 , - 2 , 3)be the given point and Q (α, β, γ) be te point on the given plane

Since PQ is parallel to the given line $\frac{x-1}{2}=\frac{y-3}{3}=\frac{z+2}{-6}$

Vector PQ is parallel to the parallel vector of the line

$\Rightarrow \frac{\alpha-1}{2}=\frac{\beta-3}{3}=\frac{\gamma+2}{-6}=\lambda$

$\Rightarrow \alpha = 2\lambda + 1 , \beta = 3\lambda - 2$ and $\gamma = -6\lambda + 3$

Now Q (α, β, γ) lie on the given plane

$\Rightarrow \alpha - \beta + \gamma = 5$

$\Rightarrow 2\lambda + 1 - 3\lambda + 2 - 6\lambda + 3 = 5$

$\Rightarrow \lambda = \frac{1}{7}$

$\Rightarrow \alpha = \frac{9}{7} , \beta = -\frac{11}{7}$ and $\gamma = \frac{15}{7}$

Therefore required distance PQ = $\sqrt{\left(\frac{9}{7}-1\right)^2+\left(\frac{-11}{7}+2\right)^2+\left(\frac{15}{7}-3\right)^2} = 1\ unit$.

Q10. Let the equation of the required plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Then the coordinates of A , B , C are A(a , 0, 0), B(0 , b, 0) and C(0 , 0, c) respectively .

So , the centoid of the triangle ABC is $(\frac{x}{a}, \frac{y}{b}, \frac{z}{c})$. But the coordinates of the centroid are (α, β, γ)

as given in the question.

$\therefore \alpha = \frac{a}{3}, \beta = \frac{b}{3}$ and $\gamma = \frac{c}{3} \Rightarrow a = 3\alpha, b = 3\beta, c = 3\gamma$.

Substituting the values of a,b, c in the above equation , we obtain the required equqtion of the plane as follows:

$$\frac{x}{3\alpha} + \frac{y}{3\beta} + \frac{z}{3\gamma} = 1 \Rightarrow \frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3.$$

HOTS

Q1. Find the angle between the vectors $\hat{i} - \hat{j}$ and $\hat{j} - \hat{k}$

Q2. The two vectors $\hat{j} + \hat{k}$ and $3\hat{i} - \hat{j} + 4\hat{k}$ represent the two side vectors AB and AC respectively of the triangle ABC . Find the length of the median through A.

Q3. Show that the points $A(1,-2,-8)$, $B(5,0,-2)$ and $C(11,3,7)$ are collinear , and find the ration in which B divides AC.

Q4. Show that thefour points with position vectors :

$4\hat{i} + 8\hat{j} + 12\hat{k}, 2\hat{i} + 4\hat{j} + 6\hat{k}, 3\hat{i} + 5\hat{j} + 4\hat{k}$ and $5\hat{i} + 8\hat{j} + 5\hat{k}$ are coplanar.

Q5. If the vertices A , B and C of a triangle ABC are (1 , 2, 3,) , (-1 , 0 , 0) and (0 , 1 , 2) respectively , then find angle ABC.

Q6. The two adajacent sides of a parallelogram are $2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\hat{i} - 2\hat{j} - 3\hat{k}$. Find the unit vector parallel to its diagonal . Also , find its area .

Q7. If $\vec{a}, \vec{b}, \vec{c}$ determine the vertices of a triangle , show that $\frac{1}{2}[\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}]$ gives the area of the triangle . Hence , deduce the condition that the three points $\vec{a}, \vec{b}, \vec{c}$ are collinear . Also, find the unit vector normal to the plane of the triangle .

Q8. If $\vec{a} + \vec{b} + \vec{c} = 0$, $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$, find the angle between $\vec{a}$ and $\vec{b}$.

Q9. Show that area of the parallelogram whose diagonals are given by $\vec{a}$ and $\vec{b}$ is $\frac{|\vec{a} \times \vec{b}|}{2}$. Also, find the area of the parallelogram whose diagonals are $2\vec{i} - \vec{j} + \vec{k}$ and $\vec{i} + 3\vec{j} - \vec{k}$.

Q10. If the vector $-\vec{i} + \vec{j} - \vec{k}$ bisects the angle between the vector $\vec{c}$ and the vectors $3\hat{i} + 4\hat{j}$, then find the unit vector in the direction of $\vec{c}$.

ANSWERS AND HINTS TO DIFFICULT QUESTIONS

Q1. $\theta = \frac{2\pi}{3}$

Q2. $BC = BA + AC = 3\hat{i} - 2\hat{j} + 3\hat{k}$

$\overrightarrow{BD} = \frac{1}{2}\overrightarrow{BC} = \frac{3}{2}\hat{i} - \hat{j} + \frac{3}{2}\hat{k}$

$Now,\ \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \frac{3}{2}\hat{i} + \frac{5}{2}\hat{j}$

$\therefore\ Length\ of\ AD = \left|\overrightarrow{AD}\right| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2} = \frac{\sqrt{34}}{2}\ units$

Q.3 Point B divides AC in the Ratio 2 : 3 .

Q4. Let four points with position vectors

$4\hat{i} + 8\hat{j} + 12\hat{k},\ 2\hat{i} + 4\hat{j} + 6\hat{k},\ 3\hat{i} + 5\hat{j} + 4\hat{k}\ and\ 5\hat{i} + 8\hat{j} + 5\hat{k}$

Be A , B , C and D respectively.

Now , $AB = -2\hat{i} - 4\hat{j} - 6\hat{k},\ AC = -\hat{i} - 3\hat{j} - 8\hat{k}\ and\ AD = \hat{i} + 0\hat{j} - 7\hat{k}$

$[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = \begin{vmatrix} -2 & -4 & -6 \\ -1 & -3 & -8 \\ 1 & 0 & -7 \end{vmatrix}$ = -2 (21 – 0) + 4 (7 + 8) - 6 (0 + 3) = -42 + 60 – 18 = 0

$\Rightarrow AB,\ AC\ and\ AD$ vectors are coplanar

Hence , points A , B , C and D are coplanar.

Q5. Angle ABC is the angle between the vectors $BA\ and\ BC$

$BA = 2\hat{i} + 2\hat{j} + 3\hat{k}\ and\ BC = \hat{i} + \hat{j} + 2\hat{k}$

$\overline{BA}\cdot\overline{BC} = \left|\overline{BA}\right|\cdot\left|\overline{BC}\right|\cos\angle ABC$

$\Rightarrow \angle ABC = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right).$

Q6. Area of the parallelogram is $11\sqrt{5}$ sq. units.

Q7. $\hat{n} = \frac{\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}}{[\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}]}.$

Q8. We have $\vec{a}+\vec{b}+\vec{c}=0$

$\Rightarrow \vec{a}+\vec{b}=-\vec{c}$

Squaring both sides , we obtain

$(\vec{a}+\vec{b})(\vec{a}+\vec{b})=(-\vec{c}).(-\vec{c})$

$|\vec{a}|^2+|\vec{b}|^2+2\vec{a}.\vec{b}=|\vec{c}|^2$

$9+25+2\times 3\times 5\cos\theta=49$

$30\cos\theta=49-34$

$\cos\theta=\frac{1}{2}$

$\theta=\frac{\pi}{3}$

Q9. Area of the parallelogram = $\frac{1}{2}\sqrt{62}$.

Q10. Let $x\hat{i}+y\hat{j}+z\hat{k}$ *be the unit vector along* $\vec{c}$.

Since $-\hat{i}+\hat{j}-\hat{k}$ bisects the angle between c and $3\hat{i}+4\hat{j}$.

$\lambda(-\hat{i}+\hat{j}-\hat{k})=(x\hat{i}+y\hat{j}+z\hat{k})\frac{3\hat{i}+4\hat{j}}{5}$

$Now,\ x^2+y^2+z^2=1\Rightarrow\left(-\lambda-\frac{3}{5}\right)+\left(\lambda-\frac{4}{5}\right)+\lambda^2=1$

$\Rightarrow\lambda=0\ or\ \lambda=\frac{2}{15}$ But $\lambda\neq 0$ because $\lambda=0$ implies that the given vectors are parallel

$\lambda=\frac{2}{15}\Rightarrow x=-\frac{11}{15},\ y=-\frac{10}{15},\ z=-\frac{2}{15}$

$x\hat{i}+y\hat{j}+z\hat{k}=-\frac{1}{15}(11\hat{i}+10\hat{j}+2\hat{k})$

ADDITIONAL IMPORTANT BOARD QUESTIONS

1. If $\vec{a},\vec{b},\vec{c}$ are the unit vectors such that $\vec{a}.\vec{b}=\vec{a}.\vec{c}=0$ and the angle between $\vec{b}$ *and* $\vec{c}$ *is* $\frac{\pi}{6}$, then prove that (i) $\vec{a}=\pm 2(\vec{b}\times\vec{c})$ (ii) $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]=\pm 1$.
2. Show that the vectors $\vec{a},\vec{b}$ *and* $\vec{c}$ are coplanar , if $\vec{a}+\vec{b},\vec{b}+\vec{c},\vec{c}+\vec{a}$ are coplanar.
3. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}-5\hat{k}$ *and* $2\hat{i}+2\hat{j}+3\hat{k}$. Find the two unit vectors parallel to its diagonals . Using the diagonal vectors , find the area of parallelogram.

4. Find the angle between the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, if $\vec{a}=2\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}+\hat{j}-2\hat{k}$ and hence ,find a vector perpendicular to both $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.
5. Show that the four points A (4 , 5 , 1) , B (0 , -1 , -1), C (3 , 9 , 4) and D (-4 , 4 , 4) are coplanar.
6. Given that vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ form a triangle such that $\vec{a}=\vec{b}+\vec{c}$. Find p, q , r , s such that area of triangle is $5\sqrt{6}$, where $\vec{a}=p\hat{i}+q\hat{j}+r\hat{k}, \vec{b}=s\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}-2\hat{k}$.
7. If $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$ and $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$, show that $\vec{a}-\vec{d}$ is parallel to $\vec{b}-\vec{c}$, where $\vec{a}\neq\vec{d}$ and $\vec{b}\neq\vec{c}$.
8. Prove that , for any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$: $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]=2[\vec{a}\ \vec{b}\ \vec{c}]$.
9. Find a unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$.
10. If $\vec{\alpha}=3\hat{i}+4\hat{j}+5\hat{k}$ and $\vec{\beta}=2\hat{i}+\hat{j}-4\hat{k}$, then express $\vec{\beta}$ in the form $\vec{\beta}=\vec{\beta}_1+\vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

SOLUTIONS

3. Unit vector parallel to $\vec{d}_1=\frac{1}{2\sqrt{6}}\left[4\hat{i}-2\hat{j}-2\hat{k}\right]$, Unit vector parallel to $\vec{d}_2=\frac{1}{10}\left[6\hat{j}+8\hat{k}\right]$.

Area of parallelogram = $2\sqrt{101}$ sq. units.

4. $\theta=\frac{\pi}{2}$, $2\hat{i}-26\hat{j}-10\hat{k}$

6. p = - 8 , 8 ; q = 4 ; r= 2 and s = - 11 , 5.

9. $\pm\left(\frac{2}{3}\hat{i}-\frac{2}{3}\hat{j}-\frac{1}{3}\hat{k}\right)$.

10. $\vec{\beta}_1=\frac{-3}{5}\hat{i}-\frac{4}{5}\hat{j}-\hat{k}$, $\vec{\beta}_2=\frac{13}{5}\hat{i}+\frac{9}{5}\hat{j}-3\hat{k}$, $\vec{\beta}=2\hat{i}+\hat{j}-4\hat{k}$.