Very Short Answer Questions (PYQ

[1 Mark]

Q.1. For what value of *x*, the following matrix is singular?

$$\begin{bmatrix} 5-x & x+1 \\ 2 & 4 \end{bmatrix}$$

Ans.

Let $\begin{vmatrix} 5-x & x+1 \\ 2 & 4 \end{vmatrix} = 0$

For A to be singular, $|A| = 0$

$\Rightarrow \quad \begin{vmatrix} 5-x & x+1 \\ 2 & 4 \end{vmatrix} = 0$

$\Rightarrow \quad 4(5-x) - 2(x+1) = 0 \quad \Rightarrow \quad 20 - 4x - 2x - 2 = 0$

$\Rightarrow \quad 18 = 6x \qquad \Rightarrow \quad x = 3.$

Q.2. Let *A* be a square matric of order 3 × 3. Write the value of |2*A*|, where |*A*| = 4.

Ans.

$\because \quad |2A| = 2^n\,|A|$, where n is order of matrix A.

Here $\quad |A| = 4$ and $n = 3$

$|2A| = 2^3 \times 4 = 32$

Q.3. If $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$**, then write the value of *x*.**

Ans.

Given $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix}$

$\Rightarrow \quad (x+1)(x+2) - (x-1)(x-3) = 12 + 1$

$\Rightarrow \quad x^2 + 2x + x + 2 - x^2 + 3x + x - 3 = 13$

$\Rightarrow \quad 7x - 1 = 13$

$\Rightarrow \quad 7x = 14$

$\Rightarrow \quad x = 2$

Q.4. Evaluate: $\begin{vmatrix} a+ & c+ \\ -c+ & a- \end{vmatrix}$

Ans.

$(a+ib)(a-ib) - (c+id)(-c+id) = [a^2 - i^2b^2] - [i^2d^2 - c^2]$

$= (a^2 + b^2) - (-d^2 - c^2) \qquad [\because \ i^2 = -1]$

$= a^2 + b^2 + d^2 + c^2$

$= a^2 + b^2 + c^2 + d^2$

Q.5. Find the cofactor of a_{12} in the following:

$$\begin{vmatrix} 1 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$$

Ans.

$a_{12} = (-1)^{1+2} . M_{12}$

$= - \begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix}$

$= -(-42 - 4) = 46$

Q.6. If $\begin{vmatrix} x+2 & 3 \\ x+5 & 4 \end{vmatrix} = 3,$ **, then find the value of *x*.**

Ans.

Here, $\begin{vmatrix} x+2 & 3 \\ x+5 & 4 \end{vmatrix} = 3$

$\Rightarrow \quad 4x + 8 - 3x - 15 = 3$

$\Rightarrow \quad x - 7 = 3 \quad \Rightarrow \quad x = 10$

Q.7. Write the value of the following determinant:

$$\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$$

Ans.

$$\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix} = 3x\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 2 & 3 & 4 \end{vmatrix} \qquad [\text{Taking out } 3x \text{ common from } R_3]$$

$$= 3x \times 0 = 0 \qquad [\because R_1 = R_3]$$

Q.8. Write the value of the following determinant:

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

Ans.

Applying $C_1 \rightarrow C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} = 0 \qquad [\because \text{All elements of } C_1 \text{ are zero}]$$

Q.9. Find the value of *x* from the following:

$$\begin{vmatrix} x & 4 \\ 2 & 2x \end{vmatrix} = 0$$

Ans.

Here $\begin{vmatrix} x & 4 \\ 2 & 2x \end{vmatrix} = 0$

$\Rightarrow \quad 2x^2 - 8 = 0 \quad \Rightarrow \quad x^2 = 4$

$\Rightarrow \quad x = \pm 2$

Q.10. What is the value of the determinant $\begin{vmatrix} 0 & 2 & 0 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix}$**?**

Ans.

Let $\Delta = \begin{vmatrix} 0 & 2 & 0 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix} = 0(18 - 20) - 2(12 - 16) + 0(10 - 12) = 8$

Q.11. Show that the points (1, 0), (6, 0), (0, 0) are collinear.

Ans.

Since $\begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0$

Hence (1, 0), (6, 0) and (0, 0) are collinear.

Q.12. What positive value of *x* makes the following pair of determinants equal?

$$\begin{vmatrix} 2x & 3 \\ 5 & x \end{vmatrix}, \begin{vmatrix} 16 & 3 \\ 5 & 2 \end{vmatrix}$$

Ans.

$\because \qquad \begin{vmatrix} 2x & 3 \\ 5 & x \end{vmatrix} = \begin{vmatrix} 16 & 3 \\ 5 & 2 \end{vmatrix}$

$\Rightarrow \qquad 2x^2 - 15 = 32 - 15 \qquad \Rightarrow \qquad 2x^2 = 32$

$\Rightarrow \qquad x^2 = 16 \qquad \Rightarrow \qquad x = \pm 4$

$\Rightarrow \qquad x = 4$ (+ve value).

Q.13. Find the minor of the element of second row and third column (a_{23}) in the following determinant:

$$\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$$

Ans.

We have,

$$\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$$

Minor of an element $a_{23} = M_{23} = \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix} = 10 + 3 = 13$

Q.14. Evaluate: $\begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix}$

Ans. Expanding the determinant, we get

$\cos 15°. \cos 75° – \sin 15°. \sin 75° = \cos (15° + 75°) = \cos 90° = 0$

[**Note:** $\cos (A + B) = \cos A. \cos B – \sin A. \sin B$]

Q.15. If $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$**, then write the minor of the element a_{23}.**

Ans. $\text{Minor of } a_{23} = M_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7.$

Q.16. Write the value of the following determinant:

$$\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

Ans.

$$\text{Let} \quad \Delta = \begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

Applying $R_1 \to R_1 - 6R_3$, we get

$$\Delta = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 0 \qquad [\because R_1 \text{ is zero}]$$

Q.17. If A_{ij} is the cofactor of the element a_{ij} of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$, then write the value of a_{32}. A_{32}.

Ans.

$$a_{32}.A_{32} = 5 \times (-1)^{3+2} \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix}$$

$$= -5\ (8 - 30) = -5 \times -22 = 110$$

Q.18. If A is a square matrix and $|A| = 2$, then write the value of $|AA'|$, where $|A'|$ is the transpose of matrix A.

Ans. |AA′|=|A|.|A′|=|A|.|A|=|A|2=22=4

[**Note:** |A|.|B| and |A|=||AT||, where *A* and *B* are square matrices.]

Very Short Answer Questions (OIQ)

[1 Mark]

Q.1. Write the value of $\begin{vmatrix} \sin 20^\circ & -\cos 20^\circ \\ \sin 70^\circ & \cos 70^\circ \end{vmatrix}$.

$$\begin{vmatrix} \sin 20^\circ & -\cos 20^\circ \\ \sin 70^\circ & \cos 70^\circ \end{vmatrix} = \sin 20^\circ . \cos 70^\circ + \cos 20^\circ . \sin 70^\circ$$

Ans. $= \sin(20^\circ + 70^\circ) = \sin 90^\circ = 1.$

Q.2. If A is square matrix of order 3 such that $|A| = \lambda$, then write the value of $|-A|$.

Ans. $\because$ $|A| = \lambda$ and order of $A = 3$

$\therefore$ $|-A| = (-1)^3 .|A| = -1 \times \lambda = -\lambda$

Q.3. Find the value of $\begin{vmatrix} \sin A & -\sin B \\ \cos A & \cos B \end{vmatrix}$ **where $A = 53°$, $B = 37°$.**

Ans.

Let $\Delta = \begin{vmatrix} \sin A & -\sin B \\ \cos A & \cos B \end{vmatrix}$

$\Delta = \sin A \cos B + \sin B \cos A \quad \Rightarrow \quad \Delta = \sin(A + B)$

$\Delta = \sin(53 + 37) = \sin 90^\circ = 1$

Short Answer Questions-I (PYQ)

[2 Mark]

Q.1. If $A=\begin{bmatrix}1 & 2\\ 4 & 2\end{bmatrix}$, then find the value of *k* if |2*A*| = *k*|*A*|.

Ans.

$$\because \quad A=\begin{bmatrix}1 & 2\\ 4 & 2\end{bmatrix} \quad \Rightarrow \quad 2A=\begin{bmatrix}2 & 4\\ 8 & 4\end{bmatrix}$$

Given, $|2A| = k|A|$

$$\Rightarrow \quad \begin{vmatrix}2 & 4\\ 8 & 4\end{vmatrix} = k\begin{vmatrix}1 & 2\\ 4 & 2\end{vmatrix}$$

$$\Rightarrow \quad 8-32 = k\{2-8\} \quad \Rightarrow \quad -24 = -6k \quad \Rightarrow \quad k = 4$$

Q.2. What is the value of the following determinant?

Ans.

$$\Delta = \begin{vmatrix}4 & a & b+c\\ 4 & b & c+a\\ 4 & c & a+b\end{vmatrix}$$

Q.3. If $\begin{vmatrix}x & x\\ 1 & x\end{vmatrix} = \begin{vmatrix}3 & 4\\ 1 & 2\end{vmatrix}$, then write the positive value of *x*.

Ans.

We have $\begin{vmatrix}x & x\\ 1 & x\end{vmatrix} = \begin{vmatrix}3 & 4\\ 1 & 2\end{vmatrix}$

$$\Rightarrow \quad x^2 - x = 6 - 4 \quad \Rightarrow \quad x^2 - x - 2 = 0$$

$$\Rightarrow \quad x^2 - 2x + x - 2 = 0 \quad \Rightarrow \quad x(x-2) + 1(x-2) = 0$$

$$\Rightarrow \quad (x-2)(x+1) = 0 \quad \Rightarrow \quad x = 2 \quad \text{or} \quad x = -1 \quad \text{(Not accepted)}$$

$$\Rightarrow \quad x = 2$$

Q.4. Write the value of $\Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix}$.

Ans.

Here, $\Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix}$

Applying $R_1 \to R_1 + R_2$, we get

$$= \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix}$$

Taking $(x+y+z)$ common from R_1, we get

$$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix}$$

Applying $R_3 \to R_3 + 3R_1$, we get

$$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 0 & 0 & 0 \end{vmatrix}$$

$= 0$ $\qquad [\because R_3 \text{ is zero}]$

Short Answer Questions-I (OIQ)

[2 Mark]

Q.1. Evaluate the determinant:

$$\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$$

Ans.

Let $\Delta = \begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$

$= (x+1)\begin{vmatrix} x^2 - x + 1 & x - 1 \\ 1 & 1 \end{vmatrix}$ [Taking out $(x+1)$ common from R_2]

$= (x+1)\{x^2 - x + 1 - x + 1\} = (x+1)(x^2 - 2x + 2)$

$= x^3 - 2x^2 + 2x + x^2 - 2x + 2 = x^3 - x^2 + 2$

Q.2. Evaluate: $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$

Ans.

$$\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix}$$

$= 0 + 2\begin{vmatrix} a & b & c \\ x & y & z \\ x & y & z \end{vmatrix}$ [$\because$ Two rows are same, so determinant is zero]

$= 0 + 2 \times 0 = 0$

Q.3. Evaluate: $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

Ans.

$$\text{Let } \Delta = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

Applying $C_2 \to C_2 + C_3$, we get

$$= \begin{vmatrix} 1 & a+b+c & b+c \\ 1 & a+b+c & c+a \\ 1 & a+b+c & a+b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix}$$

$= (a+b+c) . 0$ [$\because$ Two columns are same, so determinant is zero]

$= 0$

Q.4. What is the value of the determinant given below?

$$\begin{vmatrix} 6 & a & b+c \\ 6 & b & c+a \\ 6 & c & a+b \end{vmatrix}$$

Ans.

$$\text{Let } \Delta = \begin{vmatrix} 6 & a & b+c \\ 6 & b & c+a \\ 6 & c & a+b \end{vmatrix}$$

Applying $C_2 \to C_2 + C_3$, we get

$$= \begin{vmatrix} 6 & a+b+c & b+c \\ 6 & a+b+c & c+a \\ 6 & a+b+c & a+b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 6 & 1 & b+c \\ 6 & 1 & c+a \\ 6 & 1 & a+b \end{vmatrix} = 6(a+b+c) \begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & (a+b) \end{vmatrix}$$

$= 6\ (a+b+c) . 0 = 0$ [$\because$ Two columns are same, so determinant is zero]

Q.5. Show that points *A* (*a*, *b* + *c*), *B* (*b*, *c* + *a*) and *C* (*c*, *a* + *b*) are collinear.

Ans.

Obviously, if the area of DABC formed by points A, B and C is zero then A, B, C will be collinear.

Now, Area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$

$= \frac{1}{2}\begin{vmatrix} a+b+c & b+c & 1 \\ a+b+c & c+a & 1 \\ a+b+c & a+b & 1 \end{vmatrix}$ [Applying $C_1 \rightarrow C_1 + C_2$]

$= \frac{a+b+c}{2}\begin{vmatrix} 1 & b+c & 1 \\ 1 & c+a & 1 \\ 1 & a+b & 1 \end{vmatrix} = \frac{a+b+c}{2} \times 0 = 0$ [$\because C_1 = C_3$]

Q.6. Evaluate: $\begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$

Ans.

$\begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$

$= 0 - \sin\alpha\,\{0 - \sin\beta.\cos\alpha\} - \cos\alpha\,\{\sin\alpha\sin\beta - 0\}$

$= \sin\alpha.\cos\alpha\sin\beta - \sin\alpha.\sin\beta.\cos\alpha$

$= 0$

Q.7. Find the area of the triangle with vertices at the points (2, 7), (1, 1), (10, 8).

Ans.

We know that area of a triangle with vertices at (x_1, y_1), (x_2, y_2), (x_3, y_3) is absolute value of

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\therefore \text{Required area} = \frac{1}{2}\begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix} = \frac{1}{2}[2(1-8) - 7(1-10) + 1(8-10)]$$

$$= \frac{1}{2}[2 \times (-7) - 7 \times (-9) + (-2)]$$

$$= \frac{1}{2}[-14 + 63 - 2] = \frac{47}{2} \text{ sq unit.}$$

Q.8. Find the value of *k*, if area of a triangle is 4 sq unit when its vertices are (*k*, 0), (4, 0) and (0, 2).

Ans.

$$\text{We know that area of triangle} = \frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix}$$

$$\Rightarrow \quad = \frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \pm 4 \qquad \text{[Given]}$$

$$\Rightarrow \quad = \frac{1}{2}[k(0-2) - 0 + 1(8-0)] = \pm 4 \quad \Rightarrow \quad \frac{1}{2}[-2k + 8] = \pm 4$$

$$\Rightarrow \quad [-k + 4] = \pm 4 \quad \Rightarrow \quad -k + 4 = 4 \quad \text{or} \quad -k + 4 = -4$$

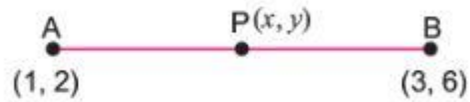
i.e., $k = 0$ or $k = 8$

Q.9. Find equation of line joining (1, 2) and (3, 6) using determinants.

Ans.

Let $P(x, y)$ be the general point on the line joining $A(1, 2)$ and $B(3, 6)$.

From figure it is obvious that the area of ΔAPB is zero.



$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 1 & 2 & 1 \\ 3 & 6 & 1 \end{vmatrix} = 0 \qquad \Rightarrow \quad \begin{vmatrix} x & y & 1 \\ 1 & 2 & 1 \\ 3 & 6 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad x(2-6) - y(1-3) + 1(6-6) = 0 \quad \Rightarrow \quad -4x + 2y = 0$$

It is required equation of line.

Q.10. Without expanding evaluate the determinant:

$$\begin{vmatrix} (a^x + a^{-x})^2 & (a^x - a^{-x})^2 & 1 \\ (a^y + a^{-y})^2 & (a^y - a^{-y})^2 & 1 \\ (a^z + a^{-z})^2 & (a^z + a^{-z})^2 & 1 \end{vmatrix}, \text{ where } a > 0 \text{ and } x, y, z \in R.$$

Ans.

Let Δ be the given determinant. Applying $C_1 \to C_1 - C_2$, we get

$$\Delta = \begin{vmatrix} 4 & (a^x - a^{-x})^2 & 1 \\ 4 & (a^y - a^{-y})^2 & 1 \\ 4 & (a^z - a^{-z})^2 & 1 \end{vmatrix} \qquad [\text{Using } (a+b)^2 - (a-b)^2 = 4ab]$$

Taking out 4 from C_1, we get

$$\Delta = 4\begin{vmatrix} 1 & (a^x - a^{-x})^2 & 1 \\ 1 & (a^y - a^{-y})^2 & 1 \\ 1 & (a^z - a^{-z})^2 & 1 \end{vmatrix} \quad \Rightarrow \quad \Delta = 4 \times 0 = 0. \qquad [\because C_1 \text{ and } C_2 \text{ are identical}]$$

Long Answer Questions (PYQ)

[4 Mark / 6 Mark]

Q.1. Using properties of determinants, prove that

$$\begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix} = (x+y+z)^3.$$

Ans.

$$\text{LHS} \quad \Delta = \begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$$

Applying $R_2 \leftrightarrow R_3$, then $R_1 \leftrightarrow R_2$, we have

$$\Delta = \begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 + R_2 + R_3$, we have

$$\Delta = \begin{vmatrix} x+y+z & y+z+x & z+x+y \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}$$

Taking out $(x+y+z)$ from first row, we have

$$\Delta = (x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 - C_3$ and $C_2 \rightarrow C_2 - C_3$, we have

$$\Delta = (x+y+z) \begin{vmatrix} 0 & 0 & 1 \\ 0 & (y+z+x) & 2y \\ (x+y+z) & z+x+y & z-x-y \end{vmatrix}$$

Expanding along first row, we have

$$\Delta = (x+y+z)\ (x+y+z)^2 = (x+y+z)^3 = \text{RHS}$$

Q.2. Prove that: $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$

Ans.

$$\text{LHS } \Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix}$$

$$\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma \end{vmatrix} \quad [\text{Applying } R_3 \to R_1 + R_3]$$

$$= (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \end{vmatrix} \quad [\text{Taking out } (\alpha+\beta+\gamma) \text{ from } R_3]$$

$$= (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta-\alpha & \gamma-\alpha \\ \alpha^2 & \beta^2-\alpha^2 & \gamma^2-\alpha^2 \\ 1 & 0 & 0 \end{vmatrix} \quad [\text{Applying } C_1 \to C_2 - C_1, C_3 \to C_3 - C_1]$$

$$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} \alpha & 1 & 1 \\ \alpha^2 & \beta+\alpha & \gamma+\alpha \\ 1 & 0 & 0 \end{vmatrix}$$

Taking out $(\beta - \alpha)$ and $(\gamma - \alpha)$ from C_2 and C_3 respectively

$$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)1.\begin{vmatrix} 1 & 1 \\ \beta+\alpha & \gamma+\alpha \end{vmatrix} \quad [\text{Expanding along } R_3]$$

$$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)(\gamma+\alpha-\beta-\alpha)$$

$$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)(\gamma-\beta) = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma) = \text{RHS}$$

Q.3. Show that: $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0,$ **where *a*, *b*, *c* are in AP.**

Ans.

$$\Delta = \begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$$

$\because$ a, b, c are in AP

$\therefore$ $2b = a + c$

Applying $R_1 \rightarrow R_1 + R_3 - 2R_2$, we have

$$= \begin{vmatrix} 0 & 0 & 0 \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$$

$\therefore$ $\Delta = 0$ $[\because R_1 = 0]$

Q.4.

Prove that: $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3.$

OR

Prove that: $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3.$

Ans.

LHS $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

Taking $2(a+b+c)$ common from C_1, we get

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{vmatrix}$$

Taking $(a+b+c)$ common from R_2 and R_3, we get

$$= 2(a+b+c)^3 \begin{vmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1, we get

$$= 2(a+b+c)^3 [1 - 0] = 2(a+b+c)^3 = \text{RHS}$$

OR

For solution replace $a \to x$, $b \to y$ and $c \to z$ in above solution.

Q.5. Using Properties of determinants, prove the following:

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3 + b^3 + c^3 - 3\,\text{abc}$$

Ans.

Let $= \begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$

$$= \begin{vmatrix} a+b+c & b & c \\ 0 & b-c & c-a \\ 2(a+b+c) & c+a & a+b \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 + C_2 + C_3]$$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix} \quad [\text{Taking out } (a+b+c) \text{ from } C_1]$$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 0 & c+a-2b & a+b-2c \end{vmatrix} \quad [\text{Applying } R_3 \to R_3 - 2R_1]$$

Expanding along C_1, we get

$$= (a+b+c) \cdot 1.\{(b-c)(a+b-2c) - (c-a)(c+a-2b)\}$$

$$= (a+b+c)\ (ab + b^2 - 2bc - ac - bc + 2c^2 - c^2 - ac + 2bc + ac + a^2 - 2ab)$$

$$= (a+b+c)\ (a^2 + b^2 + c^2 - ab - bc - ca) = a^3 + b^3 + c^3 - 3abc$$

Q.6. Using properties of determinant, solve for *x*:

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$$

Ans.

Let $\Delta = \begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix}$

$$\Delta = \begin{vmatrix} 3a-x & a-x & a-x \\ 3a-x & a+x & a-x \\ 3a-x & a-x & a+x \end{vmatrix} \qquad [\text{Applying } C_1 \to C_1 + C_2 + C_3]$$

$$\Delta = (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 1 & a+x & a-x \\ 1 & a-x & a+x \end{vmatrix}$$

$$= (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{vmatrix} \qquad [\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

Expanding along C_1, we get

$$= (3a-x)(4x^2-0) = 4x^2(3a-x)$$

Now, given that $\Delta = 0$

Therefore, $4x^2 (3a - x) = 0 \quad \Rightarrow \quad x = 0, \quad \text{or} \quad x = 3a.$

Hence, required values of x are $x = 0, 3a$.

Q.7. Using property of determinant, prove the following:

$$\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 9b^2(a+b)$$

Ans.

$$\text{LHS} = \begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$

$$= \begin{vmatrix} 3(a+b) & 3(a+b) & 3(a+b) \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} \qquad [\text{Applying } R_1 = R_1 + R_2 + R_3]$$

$$= 3(a+b)\begin{vmatrix} 1 & 1 & 1 \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} \qquad [\text{Taking } 3(a+b) \text{ common from } R_1]$$

$$= 3(a+b)\begin{vmatrix} 0 & 0 & 1 \\ b & -b & a+b \\ b & 2b & a \end{vmatrix} \qquad [\text{Applying } C_1 \to C_1 - C_3,\ C_2 \to C_2 - C_3]$$

Expanding along R_1 we get

$$= 3(a+b)\ \{1\ (2b^2 + b^2)\} = 9b^2(a+b) = \text{RHS}$$

Q.8. By using properties of determinant, prove the following:

$$\begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix} = (5x+\lambda)(\lambda - x)^2$$

Ans.

$$\text{LHS} = \begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix}$$

$$= \begin{vmatrix} 5x+\lambda & 2x & 2x \\ 5x+\lambda & x+\lambda & 2x \\ 5x+\lambda & 2x & x+\lambda \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 + C_2 + C_3]$$

$$= (5x+\lambda)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x+\lambda & 2x \\ 1 & 2x & x+\lambda \end{vmatrix} \quad [\text{Taking out } (5x+\lambda) \text{ common from } C_1]$$

$$= (5x+\lambda)\begin{vmatrix} 1 & 2x & 2x \\ 0 & \lambda-x & 0 \\ 0 & 0 & \lambda-x \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

Expanding along C_1, we get

$$= (5x+\lambda)\ (\lambda - x)2 = \text{RHS.}$$

Q.9. Using properties of determinant, prove that:

$$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix} = a^2(a+x+y+z)$$

Ans.

$$\text{LHS} = \begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$$

$$= \begin{vmatrix} a+x+y+z & y & z \\ a+x+y+z & a+y & z \\ a+x+y+z & y & a+z \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 + C_2 + C_3]$$

$$= (a+x+y+z) \begin{vmatrix} 1 & y & z \\ 1 & a+y & z \\ 1 & y & a+z \end{vmatrix} \quad [\text{Taking out } (a+x+y+z) \text{ common from } C_1]$$

$$= (a+x+y+z) \begin{vmatrix} 0 & -a & 0 \\ 1 & a+y & z \\ 1 & y & a+z \end{vmatrix} \quad [\text{Apply } R_1 \to R_1 - R_2]$$

Expanding along R_1, we get

$$= (a+x+y+z)\{0 + a(a+z-z)\}$$

$$= a^2(a+x+y+z) = \text{RHS}$$

Q.10. Using properties of determinant, prove the following:

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$$

Ans.

$$\text{LHS} = \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}$$

$$= x^2 \begin{vmatrix} x+y & 1 & 1 \\ 5x+4y & 4 & 2 \\ 10x+8y & 8 & 3 \end{vmatrix} \qquad [\text{Taking out } x \text{ from } C_2 \text{ and } C_3]$$

$$= x^2 \begin{vmatrix} x+y & 1 & 1 \\ 3x+2y & 2 & 0 \\ 7x+5y & 5 & 0 \end{vmatrix} \qquad [\text{Applying } R_2 \rightarrow R_2 - 2R_1 \text{ and } R_3 \rightarrow R_3 - 3R_1]$$

Expanding along C_3, we get

$$x^2 \left[1 \left\{(3x+2y)\ 5 - 2\ (7x+5y)\right\} - 0 + 0\right]$$

$$= x^2 \ (15x + 10y - 14x - 10y)$$

$$= x^2 \ (x) = x^3 = \text{RHS}$$

Q.11. Using properties of determinants, prove the following:

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix} = 1$$

Ans.

Let $|A| = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$

Using the transformation $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$|A| = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & -1+p \\ 0 & 3 & -2+3p \end{vmatrix}$$

Using $R_3 \to R_3 - 3R_2$

$$\Rightarrow \quad |A| = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & -1+p \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along column C_1, we get

$$|A| = 1$$

Q.13. Prove the following using properties of determinant:

$$\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 2(3\,\text{abc} - a^3 - b^3 - c^3)$$

Ans.

LHS

$$= \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \quad [\text{Applying } R_1 \to R_1 + R_2 + R_3]$$

$$= 2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} \quad [\text{Taking } 2(a+b+c) \text{ common from } R_1]$$

$$= 2(a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ c+a & b-c & b-a \\ a+b & c-a & c-b \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 - C_1;\ C_3 \to C_3 - C_1]$$

$$= 2(a+b+c)\left[1\left(bc - b^2 - c^2 + bc - bc + ac + ab - a^2\right)\right] \quad [\text{Expanding along } R_1]$$

$$= 2(a+b+c)\left(bc + ac + ab - a^2 - b^2 - c^2\right)$$

$$= -2(a+b+c)\left(a^2 + b^2 + c^2 - ab - bc - ca\right) = -2\left(a^3 + b^3 + c^3 - 3abc\right)$$

$$= 2\left(3abc - a^3 - b^3 - c^3\right) = \text{RHS}$$

Q.15. Using properties of determinants, prove the following:

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$$

OR

$$\begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix} = (5x+\lambda)(\lambda-x)^2$$

Ans.

$$\text{LHS} = \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$$= \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} \quad [\text{Applying } R_1 \rightarrow R_1 + R_2 + R_3]$$

$$= (5x+4) \begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} \quad [\text{Taking } (5x+4) \text{ common from } R_1]$$

$$= (5x+4) \begin{vmatrix} 1 & 0 & 0 \\ 2x & 4-x & 0 \\ 2x & 0 & 4-x \end{vmatrix} \quad [\text{Applying } C_2 \rightarrow C_2 - C_1;\ C_3 \rightarrow C_3 - C_1]$$

$$= (5x+4)\ [1\ \{(4-x)2 - 0\} + 0 + 0] \quad [\text{Expanding along } R_1]$$

$$= (5x+4)\ (4-x)^2 = \text{RHS}$$

OR

Solve as above by putting λ instead of 4.

Q.20. Using properties of determinant, solve the following for *x*:

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$

Ans.

Given: $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$

$\Rightarrow \quad \begin{vmatrix} x-2 & 1 & 2 \\ x-4 & -1 & -4 \\ x-8 & -11 & -40 \end{vmatrix} = 0 \qquad$ [Applying $C_2 \to C_2 - 2C_1$ and $C_3 \to C_3 - 3C_1$]

$\Rightarrow \quad \begin{vmatrix} x-2 & 1 & 2 \\ -2 & -2 & -6 \\ -6 & -12 & -42 \end{vmatrix} = 0 \qquad$ [Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$\Rightarrow \quad (x-2)(84-72) - 1(84-36) + 2(24-12) = 0 \qquad$ [Expanding along R_1]

$\Rightarrow \quad 12x - 24 - 48 + 24 = 0 \quad \Rightarrow \quad 12x = 48 \quad \Rightarrow \quad x = 4$

Q.21. Prove, using properties of determinant:

$$\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$$

Ans.

$\text{LHS} = \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$

$= \begin{vmatrix} 3y+k & y & y \\ 3y+k & y+k & y \\ 3y+k & y & y+k \end{vmatrix} \qquad$ [Applying $C_1 \to C_1 + C_2 + C_3$]

$= (3y+k)\begin{vmatrix} 1 & y & y \\ 1 & y+k & y \\ 1 & y & y+k \end{vmatrix} \qquad$ [Taking $(3y+k)$ common from C_1]

$= (3y+k)\begin{vmatrix} 1 & y & y \\ 0 & k & 0 \\ 0 & 0 & k \end{vmatrix} \qquad$ [Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

Expanding along C_1 we get

$= (3y+k)\{1(k^2 - 0) - 0 + 0\} = (3y+k) \cdot k^2 = k^2(3y+k)$

Q.22. Using properties of determinants, prove that

$$\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

OR

$$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

Q.23. Using properties of determinants, show that Δ*ABC* is an isosceles if:

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{vmatrix} = 0$$

Ans.

We have

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A+\cos A & \cos^2 B+\cos B & \cos^2 C+\cos C \end{vmatrix} = 0$$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ \cos A - \cos C & \cos B - \cos C & 1+\cos C \\ \cos^2 A+\cos A - \cos^2 C - \cos C & \cos^2 B+\cos B - \cos^2 C - \cos C & \cos^2 C+\cos C \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ \cos A - \cos C & \cos B - \cos C & 1+\cos C \\ (\cos A - \cos C)(\cos A+\cos C+1) & (\cos B - \cos C)(\cos B+\cos C+1) & \cos^2 C+\cos C \end{vmatrix}$$

Taking common $(\cos A - \cos C)$ from C_1 and $(\cos B - \cos C)$ from C_2, we get

$$\Rightarrow (\cos A - \cos C)(\cos B - \cos C)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1+\cos C \\ \cos A+\cos C+1 & \cos B+\cos C+1 & \cos^2 C+\cos C \end{vmatrix} = 0$$

Applying $C_1 \rightarrow C_1 - C_2$, we get

$$\Rightarrow \quad (\cos A - \cos C)(\cos B - \cos C)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 1+\cos C \\ \cos A - \cos B & \cos B + \cos C + 1 & \cos^2 C + \cos C \end{vmatrix}$$

Expanding along R_1, we get

$\Rightarrow \quad (\cos A - \cos C)\ (\cos B - \cos C)\ (\cos B - \cos A) = 0$

$\Rightarrow \quad \cos A - \cos C = 0 \quad$ *i.e.*, $\cos A = \cos C$

or, $\quad \cos B - \cos C = 0 \quad$ *i.e.*, $\cos B = \cos C$

or, $\quad \cos B - \cos A = 0 \quad$ *i.e.*, $\cos B = \cos A$

$A = C$ or $B = C$ or $B = A$

Hence, ΔABC is an isosceles triangle.

Q.30. Using properties of determinants, prove the following:

$$\begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix} = (1 - a^3)^2$$

Ans.

$$\text{LHS } \Delta = \begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 1 + a^2 + a & a + 1 + a^2 & a^2 + a + 1 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix} \quad [\text{Applying } R_1 \to R_1 + R_2 + R_3]$$

$$\Delta = (1 + a + a^2) \begin{vmatrix} 1 & 1 & 1 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix} \quad [\text{Taking out } (1 + x + x^2) \text{ from first row}]$$

$$\Delta = (1 + a + a^2) \begin{vmatrix} 0 & 1 & 1 \\ a^2 - 1 & 1 & a \\ a - a^2 & a^2 & 1 \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 - C_2]$$

$$\Delta = (1 + a + a^2) \begin{vmatrix} 0 & 0 & 1 \\ a^2 - 1 & 1 - a & a \\ a - a^2 & a^2 - 1 & 1 \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 - C_3]$$

Expanding along R_1 we have

$$= (1 + a + a^2) \left[(a^2 - 1)^2 - a\,(1 - a)^2\right]$$

$$= (1 + a + a^2) \left[(a + 1)^2\,(a - 1)^2 - a\,(a - 1)^2\right]$$

$$= (1 + a + a^2)\,(a - 1)^2\left[a^2 + 1 + a\right] = (1 + a + a^2)\,(a - 1)^2\left[a^2 + 1 + a\right]$$

$$= (a - 1)^2\,(1 + a + a^2)^2 = (1 - a)^2\,(1 + a + a^2)^2$$

$$= \left[(1 - a)\,(1 + a + a^2)\right]^2 = (1 - a^3)^2 = \text{RHS}$$

Q.32. If *a* ≠ *b* ≠ *c* and $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0,$ **then using properties of determinants, prove that *a* + *b* + *c* = 0.**

Ans.

We have $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} (a+b+c) & b & c \\ (a+b+c) & c & a \\ (a+b+c) & a & b \end{vmatrix} = 0$$

Taking $(a + b + c)$ common from C_1, we get

$$(a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} = 0$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get

$$(a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} = 0$$

Expanding along C_1, we get

$$(a+b+c)\ [1\ \{(c-b)\ (b-c) - (a-c)\ (a-b)\} - 0 + 0] = 0$$

$$\Rightarrow \quad (a+b+c)\ [(c-b)\ (b-c) - (a-c)\ (a-b)] = 0$$

$$\Rightarrow \quad (a+b+c)\ [(b-c)\ (b-c) + (a-c)\ (a-b)] = 0$$

$$\Rightarrow \quad (a+b+c)\ [(b-c)^2 + (a-c)\ (a-b)] = 0$$

$$\Rightarrow \quad (a+b+c)\ [(b^2 + c^2 - 2bc + a^2 - ab - ac + bc] = 0$$

$$\Rightarrow \quad (a+b+c)\ [a^2 + b^2 + c^2 - bc - ab - ac] = 0$$

$$\Rightarrow \quad (a+b+c)\tfrac{1}{2}[2a^2 + 2b^2 + 2c^2 - 2bc - 2ab - 2ac] = 0$$

$$\Rightarrow \quad (a+b+c)\ [(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$$

$$\Rightarrow \quad (a+b+c) = 0\ [\because a \neq b \neq c \Rightarrow (a-b)^2 + (b-c)^2 + (c-a)^2 \neq 0]$$

Long Answer Questions (OIQ)

[4 Mark / 6 Mark]

Q.1. Without expanding, show that:

$$\Delta = \begin{vmatrix} \text{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \text{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix}$$

Ans.

$$\text{Given, } \Delta = \begin{vmatrix} \text{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \text{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} \text{cosec}^2\theta - \cot^2\theta - 1 & \cot^2\theta & 1 \\ \cot^2\theta - \text{cosec}^2\theta + 1 & \text{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 - C_2 - C_3]$$

$$= \begin{vmatrix} 1-1 & \cot^2\theta & 1 \\ -1+1 & \text{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} \quad [\because \text{cosec}^2\theta - \cot^2\theta = 1]$$

$$= \begin{vmatrix} 0 & \cot^2\theta & 1 \\ 0 & \text{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} = 0 \quad [\because \text{All elements of } C_1 \text{ are } 0]$$

Q.2. If *a*, *b*, *c* are real numbers, then prove that

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$$

where ω is a complex number and cube root of unity.

Ans.

Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$= \begin{vmatrix} a+b+c & b & c \\ b+c+a & c & a \\ c+a+b & a & b \end{vmatrix}$ [Applying $C_1 \to C_1 + C_2 + C_3$]

$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$ [Taking out $(a + b + c)$ from C_1]

$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}$ [Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$= (a+b+c)\begin{vmatrix} c-b & a-c \\ a-b & b-c \end{vmatrix}$ [Expanding along C_1]

$= (a + b + c) \{- (b - c)^2 - (a - c)(a - b)\}$

LHS $= - (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

Also, RHS $= - (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$= - (a + b + c)(a^2 + ab\omega^2 + ac\omega + ab\omega + b^2\omega^3 + bc\omega^2 + ac\omega^2 + bc\omega^4 + c^2\omega^3)$

$= - (a + b + c)[(a^2 + b^2 + c^2 + ab(\omega^2 + \omega) + bc(\omega^2 + \omega^4) + ca(\omega + \omega^2)]$ [$\because \omega^3 = 1$]

$= - (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) =$ LHS [$\because \omega^2 + \omega + 1 = 0$ and $\omega^4 = \omega^3 . \omega = \omega$]

Q.3. Find the equation of the line joining *A* (1, 3) and *B* (0, 0) using determinants and find *k* if *D* (*k*, 0) is a point such that the area of Δ*ABD* is 3 sq units.

Ans.

Let $P(x, y)$ be any point on the line AB. Then,

$ar(\Delta ABP) = 0$

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ x & y & 1 \end{vmatrix} = 0 \quad \Rightarrow \quad \frac{1}{2}\{1(0-y) - 3(0-x) + 1(0-0)\} = 0$$

$\Rightarrow \quad 3x - y = 0$, which is the required equation of line AB.

Now, area $(\Delta ABD) = 3$ sq units

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 3 \quad \Rightarrow \quad \begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 6$$

$\Rightarrow \quad 1(0-0) - 3(0-k) + 1(0-0) = \pm 6 \Rightarrow 3k = \pm 6 \quad \Rightarrow \quad k = \pm 2$

Q.4. In a triangle *ABC*, if

$$\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A + \sin^2 A & \sin B + \sin^2 B & \sin C + \sin^2 C \end{vmatrix} = 0,$$

then prove that Δ*ABC* is an isosceles triangle.

Ans.

$$\text{Let } \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A + \sin^2 A & \sin B + \sin^2 B & \sin C + \sin^2 C \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & \sin B - \sin A & \sin C - \sin A \\ \sin A + \sin^2 A & \sin^2 B - \sin^2 A + \sin B - \sin A & \sin^2 C - \sin^2 A + \sin C - \sin A \end{vmatrix}$$

[Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$]

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & \sin B - \sin A & \sin C - \sin A \\ \sin A + \sin^2 A & (\sin B - \sin A)(\sin B + \sin A + 1) & (\sin C - \sin A)(\sin C + \sin A + 1) \end{vmatrix}$$

$$= (\sin B - \sin A)(\sin C - \sin A)\begin{vmatrix} 1 & 0 & 0 \\ 1+\sin A & 1 & 1 \\ \sin A + \sin^2 A & \sin B + \sin A + 1 & \sin C + \sin A + 1 \end{vmatrix}$$

Expanding along R_1, we get

$(\sin B - \sin A)\ (\sin C - \sin A)\ [\sin C + \sin A + 1 - \sin B - \sin A - 1]$

$= (\sin B - \sin A)\ (\sin C - \sin A)\ (\sin C - \sin B)$

$\because \quad \Delta = 0$

$\Rightarrow \quad (\sin B - \sin A)\ (\sin C - \sin B)\ (\sin C - \sin A) = 0$

$\Rightarrow \quad \sin B - \sin A = 0$ or $\sin C - \sin B = 0$ or $\sin C - \sin A = 0$

$\Rightarrow \quad B = A$ or $C = B$ or $C = A$

$\Rightarrow \quad \Delta ABC$ is an isosceles triangle.

Q.5. Let $f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix}$, then find $\lim\limits_{t \to 0} \frac{f(t)}{t^2}$.

Ans.

Given, $f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix} = \begin{vmatrix} \cos t & t & 1 \\ 0 & -t & 0 \\ \sin t & t & t \end{vmatrix}$ [Applying $R_2 \to R_2 - 2R_3$]

$$= t\begin{vmatrix} \cos t & 1 & 1 \\ 0 & -1 & 0 \\ \sin t & 1 & t \end{vmatrix}$$

Expanding along R_2, we get

$$t\,[(-1)\ (t\cos t - \sin t)] = -t^2\cos t + t\sin t$$

$\therefore \quad \lim\limits_{t \to 0} \frac{f(t)}{t^2} = \lim\limits_{t \to 0} \frac{-t^2\cos t + t\sin t}{t^2}$

$$= \lim_{t \to 0} \left(\frac{-t^2\cos t}{t^2} + \frac{t\sin t}{t^2} \right)$$

$$= \lim_{t \to 0} \left(-\cos t + \frac{\sin t}{t}\right) = -1 + \lim_{t \to 0} \frac{\sin t}{t} = -1 + 1 = 0$$