CHAPTER 1 – Application of Matrices and Determinants - Theorem

Theorem 1.1

For every square matrix A of order n, $A(\text{adj } A) = (\text{adj } A)A = |A| I_n$.

Proof

For simplicity, we prove the theorem for $n = 3$ only.

Consider $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then, we get

$$a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = |A|, \quad a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0, \quad a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 0;$$
$$a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} = 0, \quad a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} = |A|, \quad a_{21}A_{31} + a_{22}A_{32} + a_{23}A_{33} = 0;$$
$$a_{31}A_{11} + a_{32}A_{12} + a_{33}A_{13} = 0, \quad a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23} = 0, \quad a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} = |A|.$$

By using the above equations, we get

$$A(\text{adj}A) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| I_3 \qquad \ldots (1)$$

$$(\text{adj}A)A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A| I_3, \qquad \ldots (2)$$

where I_3 is the identity matrix of order 3.

So, by equations (1) and (2), we get $A(\text{adj } A) = (\text{adj } A)A = |A| I_3$.

Note

If A is a singular matrix of order n, then $|A| = 0$ and so $A(\text{adj } A) = (\text{adj } A)A = O_n$, where O_n denotes zero matrix of order n.

Theorem 1.2

If a square matrix has an inverse, then it is unique.

Proof

Let A be a square matrix order n such that an inverse of A exists. If possible, let there be two inverses B and C of A. Then, by definition, we have $AB = BA = I_n$ and $AC = CA = I_n$.

Using these equations, we get

$$C = CI_n = C(AB) = (CA)B = I_n B = B.$$

Hence the uniqueness follows.

Notation The inverse of an A is denoted by A^{-1}.

Note

$AA^{-1} = A^{-1}A = I_n$.

Theorem 1.3

Let A be square matrix of order n. Then, A^{-1} exists if and only if A is non-singular.

Proof

Suppose that A^{-1} exists. Then $AA^{-1} = A^{-1}A = I_n$.

By the product rule for determinants, we get

$\det(AA^{-1}) = \det(A)\det(A^{-1}) = \det(A^{-1})\det(A) = \det(I_n) = 1$. So, $|A| = \det(A) \neq 0$.

Hence A is non-singular.

Conversely, suppose that A is non-singular.

Then $|A| \neq 0$. By Theorem 1.1, we get

$$A(\text{adj } A) = (\text{adj } A)A = |A| I_n.$$

So, dividing by $|A|$, we get $A\left(\frac{1}{|A|}\text{adj } A\right) = \left(\frac{1}{|A|}\text{adj } A\right)A = I_n$.

Thus, we are able to find a matrix $B = \frac{1}{|A|}\text{adj } A$ such that $AB = BA = I_n$.

Hence, the inverse of A exists and it is given by $\mathbf{A^{-1} = \frac{1}{|A|}\text{adj } A}$.

Theorem 1.4

If A is non-singular, then

(i) $|A^{-1}| = \frac{1}{|A|}$ (ii) $(A^T)^{-1} = (A^{-1})^T$ (iii) $(\lambda A)^{-1} = \frac{1}{\lambda}A^{-1}$, where λ is a non-zero scalar.

Proof

Let A be non-singular. Then $|A| \neq 0$ and A^{-1} exists. By definition,

$$AA^{-1} = A^{-1}A = I_n. \quad \ldots(1)$$

(i) By (1), we get $|AA^{-1}| = |A^{-1}A| = |I_n|$.

Using the product rule for determinants, we get $|A||A^{-1}| = |I_n| = 1$.

Hence, $|A^{-1}| = \frac{1}{|A|}$.

(ii) From (1), we get $(AA^{-1})^T = (A^{-1}A)^T = (I_n)^T$.

Using the reversal law of transpose, we get $(A^{-1})^T A^T = A^T (A^{-1})^T = I_n$. Hence $(A^T)^{-1} = (A^{-1})^T$.

(iii) Since λ is a non-zero number, from (1), we get $(\lambda A)\left(\frac{1}{\lambda}A^{-1}\right) = \left(\frac{1}{\lambda}A^{-1}\right)(\lambda A) = I_n$.

So, $(\lambda A)^{-1} = \frac{1}{\lambda}A^{-1}$. ■

Theorem 1.5 (Left Cancellation LaW)
Let $A, B,$ and C be square matrices of order n. If A is non-singular and $AB = AC$, then $B = C$.

Proof

Since A is non-singular, A^{-1} exists and $AA^{-1} = A^{-1}A = I_n$. Taking $AB = AC$ and pre-multiplying both sides by A^{-1}, we get $A^{-1}(AB) = A^{-1}(AC)$. By using the associative property of matrix multiplication and property of inverse matrix, we get $B = C$. ■

Theorem1.6 (Right Cancellation Law)
Let $A, B,$ and C be square matrices of order n. If A is non-singular and $BA = CA$, then $B = C$.

Proof

Since A is non-singular, A^{-1} exists and $AA^{-1} = A^{-1}A = I_n$. Taking $BA = CA$ and post-multiplying both sides by A^{-1}, we get $(BA)A^{-1} = (CA)A^{-1}$. By using the associative property of matrix multiplication and property of inverse matrix, we get $B = C$. ■

Note

If A is singular and $AB = AC$ or $BA = CA$, then B and C need not be equal. For instance, consider the following matrices:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

We note that $|A| = 0$ and $AB = AC$; but $B \neq C$.

Theorem 1.7 (Reversal Law for Inverses)
If A and B are non-singular matrices of the same order, then the product AB is also non-singular and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof

Assume that A and B are non-singular matrices of same order n. Then, $|A| \neq 0$, $|B| \neq 0$, both A^{-1} and B^{-1} exist and they are of order n. The products AB and $B^{-1}A^{-1}$ can be found and they are also of order n. Using the product rule for determinants, we get $|AB| = |A||B| \neq 0$. So, AB is non-singular and

$$(AB)(B^{-1}A^{-1}) = (A(BB^{-1}))A^{-1} = (AI_n)A^{-1} = AA^{-1} = I_n;$$
$$(B^{-1}A^{-1})(AB) = (B^{-1}(A^{-1}A))B = (B^{-1}I_n)B = B^{-1}B = I_n.$$

Hence $(AB)^{-1} = B^{-1}A^{-1}$.

Theorem 1.8 (Law of Double Inverse)

If A is non-singular, then A^{-1} is also non-singular and $(A^{-1})^{-1} = A$

Proof

Assume that A is non-singular. Then $|A| \neq 0$, and A^{-1} exists.

Now $|A^{-1}| = \frac{1}{|A|} \neq 0 \Rightarrow A^{-1}$ is also non-singular, and $AA^{-1} = A^{-1}A = I$.

Now, $AA^{-1} = I \Rightarrow \left(AA^{-1}\right)^{-1} = I \Rightarrow \left(A^{-1}\right)^{-1} A^{-1} = I$. ... (1)

Post-multiplying by A on both sides of equation (1), we get $\left(A^{-1}\right)^{-1} = A$.

Theorem 1.9

If A is a non-singular square matrix of order n, then

(i) $(\text{adj } A)^{-1} = \text{adj}\left(A^{-1}\right) = \frac{1}{|A|} A$

(ii) $|\text{adj } A| = |A|^{n-1}$

(iii) $\text{adj}(\text{adj } A) = |A|^{n-2} A$

(iv) $\text{adj}(\lambda A) = \lambda^{n-1}\text{adj}(A)$, λ is a nonzero scalar

(v) $|\text{adj}(\text{adj}A)| = |A|^{(n-1)^2}$

(vi) $(\text{adj } A)^T = \text{adj}\left(A^T\right)$

Proof

Since A is a non-singular square matrix, we have $|A| \neq 0$ and so, we get

(i) $A^{-1} = \frac{1}{|A|}(\text{adj } A) \Rightarrow \text{adj } A = |A| A^{-1} \Rightarrow (\text{adj } A)^{-1} = \left(|A| A^{-1}\right)^{-1} = \frac{1}{|A|} A$.

Replacing A by A^{-1} in $\text{adj } A = |A| A^{-1}$, we get $\text{adj}\left(A^{-1}\right) = \left|A^{-1}\right| \left(A^{-1}\right)^{-1} = \frac{1}{|A|} A$.

Hence, we get $(\text{adj } A)^{-1} = \text{adj}\left(A^{-1}\right) = \frac{1}{|A|} A$.

(ii) $A(\text{adj } A) = (\text{adj } A)A = |A| I_n \Rightarrow \det\left(A(\text{adj } A)\right) = \det\left((\text{adj } A)A\right) = \det\left(|A| I_n\right)$

$\Rightarrow |A||\text{adj } A| = |A|^n \Rightarrow |\text{adj } A| = |A|^{n-1}$.

(iii) For any non-singular matrix B of order n, we have $B(\text{adj } B) = (\text{adj } B)B = |B| I_n$.

Put $B = \text{adj } A$. Then, we get $(\text{adj } A)(\text{adj}(\text{adj } A)) = |\text{adj } A| I_n$.

So, since $|\text{adj } A| = |A|^{n-1}$, we get $(\text{adj } A)(\text{adj}(\text{adj } A)) = |A|^{n-1} I_n$.

Pre-multiplying both sides by A, we get $A\left((\text{adj } A)(\text{adj }(\text{adj } A))\right) = A\left(|A|^{n-1} I_n\right)$.

Using the associative property of matrix multiplication, we get

$\left(A(\text{adj } A)\right)\text{adj}(\text{adj } A) = A\left(|A|^{n-1} I_n\right)$.

Hence, we get $\left(|A| I_n\right)\left(\text{adj}(\text{adj } A)\right) = |A|^{n-1} A$. That is, $\text{adj}(\text{adj } A) = |A|^{n-2} A$.

(iv) Replacing A by λA in $\text{adj}(A) = |A| A^{-1}$, we get

$\text{adj}(\lambda A) = |\lambda A| (\lambda A)^{-1} = \lambda^n |A| \frac{1}{\lambda} A^{-1} = \lambda^{n-1}\text{adj}(A)$

(v) By (iii), we have $\text{adj}(\text{adj } A) = |A|^{n-2} A$. So, by taking determinant on both sides, we get

$|\text{adj}(\text{adj}A)| = \left||A|^{n-2} A\right| = \left(|A|^{(n-2)}\right)^n |A| = |A|^{n^2-2n+1} = |A|^{(n-1)^2}$.

(vi) Replacing A by A^T in $A^{-1} = \frac{1}{|A|}\text{adj } A$, we get $\left(A^T\right)^{-1} = \frac{1}{|A^T|}\text{adj}\left(A^T\right)$ and hence, we get $\text{adj}\left(A^T\right) = |A^T| \left(A^T\right)^{-1} = |A| \left(A^{-1}\right)^T = \left(|A| A^{-1}\right)^T = \left(|A| \frac{1}{|A|}\text{adj } A\right)^T = (\text{adj } A)^T$. ■

Note

If A is a non-singular matrix of order 3, then, $|A| \neq 0$. By property (*ii*), we get $|\text{adj} A| = |A|^2$ and so, $|\text{adj } A|$ is positive. Then, we get $|A| = \pm\sqrt{|\text{adj } A|}$.

So, we get $A^{-1} = \pm\dfrac{1}{\sqrt{|\text{adj } A|}}\,\text{adj } A$.

Further, by the property (*iii*), we get $A = \dfrac{1}{|A|}\text{adj}(\text{adj } A)$.

Hence, if A is a non-singular matrix of order 3, then, we get $A = \pm\dfrac{1}{\sqrt{|\text{adj } A|}}\,\text{adj}(\text{adj } A)$.

Theorem 1.10

If A and B are any two non-singular square matrices of order n, then

$$\text{adj}(AB) = (\text{adj } B)(\text{adj } A).$$

Proof

Replacing A by AB in $\text{adj}(A) = |A|A^{-1}$, we get

$$\text{adj}(AB) = |AB|(AB)^{-1} = \left(|B|B^{-1}\right)\left(|A|A^{-1}\right) = \text{adj}(B)\,\text{adj}(A).$$

Theorem 1.11

The rank of a matrix in row echelon form is the number of non-zero rows in it.

The rank of a matrix which is not in a row-echelon form, can be found by applying the following result which is stated without proof.

Theorem 1.12

The rank of a non-zero matrix is equal to the number of non-zero rows in a row-echelon form of the matrix.

Theorem 1.13

Every non-singular matrix can be transformed to an identity matrix, by a sequence of elementary row operations.

Theorem 1.14 (Rouche'-Capelli Theorem)

A system of linear equations, written in the matrix form as $AX = B$, is consistent if and only if the rank of the coefficient matrix is equal to the rank of the augmented matrix; that is, $\rho(A) = \rho([A\,|\,B])$.