Theorem 3.1 (The Fundamental Theorem of Algebra)

Every polynomial equation of degree $n \geq 1$ has at least one root in $\mathbb{C}$.

Theorem 3.2 (Complex Conjugate Root Theorem)

If a complex number z_0 is a root of a polynomial equation with real coefficients, then its complex conjugate $\overline{z_0}$ is also a root.

Proof

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_o = 0$ be a polynomial equation with real coefficients. Let z_0 be a root of this polynomial equation. So, $P(z_0)=0$. Now

$$\begin{aligned} P(\overline{z_0}) &= a_n \overline{z}_0^{\,n} + a_{n-1}\overline{z}_0^{\,n-1} + \cdots + a_1 \overline{z}_0 + a_0 \\ &= a_n \overline{z_0}^{\,n} + a_{n-1}\overline{z_0}^{\,n-1} + \cdots + a_1 \overline{z_0} + a_0 \\ &= \overline{a_n}\,\overline{z_0}^{\,n} + \overline{a_{n-1}}\,\overline{z_0}^{\,n-1} + \cdots + \overline{a_1}\,\overline{z_0} + \overline{a_0} \qquad (a_r = \overline{a_r} \text{ as } a_r \text{ is real for all } r) \\ &= \overline{a_n z_0^{\,n}} + \overline{a_{n-1}z_0^{\,n-1}} + \cdots + \overline{a_1 z_0} + \overline{a_0} \\ &= \overline{a_n z_0^{\,n} + a_{n-1}z_0^{\,n-1} + \cdots + a_1 z_0 + a_0} = \overline{P(z_0)} = \overline{0} = 0 \end{aligned}$$

That is $P(\overline{z}_0) = 0$; this implies that whenever z_0 is a root (i.e. $P(z_0)=0$), its conjugate $\overline{z}_0$ is also a root .

Theorem 3.3

Let p and q be rational numbers such that $\sqrt{q}$ is irrational. If $p+\sqrt{q}$ is a root of a quadratic equation with rational coefficients, then $p-\sqrt{q}$ is also a root of the same equation.

Proof

We prove the theorem by assuming that the quadratic equation is a monic polynomial equation. The result for non-monic polynomial equation can be proved in a similar way.

Let p and q be rational numbers such that $\sqrt{q}$ is irrational. Let $p+\sqrt{q}$ be a root of the equation $x^2 + bx + c = 0$ where b and c are rational numbers.

Let α be the other root. Computing the sum of the roots, we get

$$\alpha + p + \sqrt{q} = -b$$

and hence $\alpha + \sqrt{q} = -b - p \in \mathbb{Q}$. Taking $-b-p$ as s, we have $\alpha + \sqrt{q} = s$.

This implies that

$$\alpha = s - \sqrt{q}.$$

Computing the product of the roots, gives

$$(s-\sqrt{q})(p+\sqrt{q}) = c$$

and hence $(sp-q)+(s-p)\sqrt{q} = c \in \mathbb{Q}$. Thus $s-p=0$. This implies that $s=p$ and hence we get $\alpha = p - \sqrt{q}$. So, the other root is $p-\sqrt{q}$.

Theorem 3.4

Let p and q be rational numbers so that $\sqrt{p}$ and $\sqrt{q}$ are irrational numbers; further let one of $\sqrt{p}$ and $\sqrt{q}$ be not a rational multiple of the other. If $\sqrt{p}+\sqrt{q}$ is a root of a polynomial equation with rational coefficients, then $\sqrt{p}-\sqrt{q}, -\sqrt{p}+\sqrt{q}$, and $-\sqrt{p}-\sqrt{q}$ are also roots of the same polynomial equation.

Theorem 3.5 (Rational Root Theorem)

Let $a_n x^n + \cdots + a_1 x + a_0$ with $a_n \neq 0$ and $a_0 \neq 0$, be a polynomial with integer coefficients. If $\frac{p}{q}$, with $(p,q)=1$, is a root of the polynomial, then p is a factor of a_0 and q is a factor of a_n.

Theorem 3.6

A polynomial equation $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0 = 0$, $(a_n \neq 0)$ is a reciprocal equation if, and only if, one of the following two statements is true:

(i) $a_n = a_0, \quad a_{n-1} = a_1, \quad a_{n-2} = a_2 \cdots$

(ii) $a_n = -a_0, \; a_{n-1} = -a_1, \; a_{n-2} = -a_2, \cdots$

Proof

Consider the polynomial equation

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0 = 0. \qquad \ldots (1)$$

Replacing x by $\frac{1}{x}$ in (1), we get

$$P\left(\frac{1}{x}\right) = \frac{a_n}{x^n} + \frac{a_{n-1}}{x^{n-1}} + \frac{a_{n-2}}{x^{n-2}} + \cdots + \frac{a_2}{x^2} + \frac{a_1}{x} + a_0 = 0. \qquad \ldots (2)$$

Multiplying both sides of (2) by x^n, we get

$$x^n P\left(\frac{1}{x}\right) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n = 0. \qquad \ldots (3)$$

Now, (1) is a reciprocal equation $\Leftrightarrow$ $P(x) = \pm\, x^n P\left(\frac{1}{x}\right) \Leftrightarrow$ (1) and (3) are same.

This is possible $\Leftrightarrow$ $\frac{a_n}{a_0} = \frac{a_{n-1}}{a_1} = \frac{a_{n-2}}{a_2} = \cdots = \frac{a_2}{a_{n-2}} = \frac{a_1}{a_{n-1}} = \frac{a_0}{a_n}$.

Let the proportion be equal to λ. Then, we get $\frac{a_n}{a_0} = \lambda$ and $\frac{a_0}{a_n} = \lambda$. Multiplying these equations, we get $\lambda^2 = 1$. So, we get two cases $\lambda = 1$ and $\lambda = -1$.

Case (i) :

$\lambda = 1$ In this case, we have $a_n = a_0, \; a_{n-1} = a_1, \; a_{n-2} = a_2, \; \cdots$.

That is, the coefficients of (1) from the beginning are equal to the coefficients from the end.

Case (ii) :

$\lambda = -1$ In this case, we have $a_n = -a_0, \; a_{n-1} = -a_1, \; a_{n-2} = -a_2, \; \cdots$.

That is, the coefficients of (1) from the beginning are equal in magnitude to the coefficients from the end, but opposite in sign.

Theorem 3.7 (Descartes Rule)

If p is the number of positive zeros of a polynomial $P(x)$ with real coefficients and s is the number of sign changes in coefficients of $P(x)$, then $s-p$ is a nonnegative even integer.